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"Asset pricing under optimal contracts"
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Abstract

We consider the problem of finding equilibrium asset prices in a financial market in which a portfolio manager (Agent) invests on behalf of an investor (Principal), who compensates the manager with an optimal contract. We extend a model from Buffa, Vayanos and Woolley (2014), BVW (2014), by allowing general contracts. We find that the optimal contract rewards Agent for taking specific risk of individual assets and not only the systematic risk of the index by using the quadratic variation of the deviation between the portfolio return and the return of an index portfolio. Similarly, to BVW (2014), we find that, in equilibrium, the stocks in large supply have high-risk premia, while the stocks in low supply have low risk premia, and this effect is stronger as agency friction increases. However, by using our risk-incentive optimal contract, the sensitivity of the price distortion to agency frictions is of an order of magnitude smaller compared to the price distortion in BVW (2014), where only contracts linear in portfolio value and index are allowed.

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Asset pricing under optimal contracts [*]

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Keywords: asset-management, equilibrium asset pricing, optimal contracts, principal–agent problem.
2000 Mathematics Subject Classification: 91B40, 93E20
JEL classification: C61, C73, D82, J33, M52

1   Introduction

We consider the problem of asset pricing with delegated portfolio management, that is, of finding asset prices so that the financial market is in equilibrium when the portfolio managers are offered optimal compensation contracts. The fact that an increasing percentage of investment funds is run by investment managers underlines the importance of studying the effect of managerial actions

on asset prices. Thus, the problem is important, however, it is also difficult. There are extensive studies that consider various equilibrium models of asset prices, but, partly due to technical difficulties, there are almost no results where asset pricing is combined with optimal contracting between portfolio managers and investors. A notable exception is Buffa et al. (2014), henceforth BVW (2014), which inspired the current paper.

BVW (2014) considers a market with three types of participants: portfolio managers who can divert a part of the managed portfolio funds to their own private savings; rational investors who can hire managers to invest on investors behalf in individual stocks, while investors can invest privately only in the index; and buy-and-hold investors. The first two types have CARA utility functions. The paper considers two models: one in which the dividends have square-root dynamics, and the other in which they have OU (Orstein-Uhlenbeck) dynamics. The representative CARA investor chooses optimally the contract to pay the representative manager, but is allowed to do so only in a subfamily of all possible contracts – those that are linear in the investor's portfolio value and the stock index. This would, indeed, be optimal in the classical moral hazard continuous-time models of Holmstrom and Milgrom (1987) and Sannikov (2008), in which the manager can only affect the return of the output process. However, when the manager can also affect the volatility of the output, as is the case in portfolio management, it was shown in Cvitanić et al. (2016a) and Cvitanić et al. (2016b), henceforth CPT (2016ab), that the optimal contract makes use also of the quadratic variation of the output and its covariations with the contractible factors. We use that insight to extend the family of admissible contracts in this paper.

In CPT (2016ab) the manager is paid only at the final time, and the model is one of partial equilibrium. In contrast, we identify optimal contracts in the model of BVW (2014), which is a full equilibrium model and on infinite horizon, by adapting to our setting the approach of CPT (2016ab). In the OU model, we find that the optimal contract is linear in the investor's portfolio value, the stock index, and the quadratic variation of the deviation of the portfolio return from the return of an index portfolio. Thus, the contract rewards Agent for taking the specific risk of individual risky assets beyond the systematic risk of the index. To the best of our knowledge, this is the first general equilibrium model in which such a contract is shown to be optimal. We show, in a numerical example, that given asset prices, the contracts that include the quadratic variation component can substantially increase investor's optimal value relative to the contracts that do not include it. The use of the quadratic variation, which, in practice, would correspond to using the sample variance, is, as noted in CPT (2016a), in the spirit of using the sample Sharpe ratio when compensating portfolio managers. However, in our model, in equilibrium, the principal rewards the agent for higher values of the quadratic variation, rather than penalizing him, to provide proper incentives for risk-taking beyond solely taking the risk of the index.

We leave the square-root model for future research. The difficulty with the square-root model is that the linear contracts with constant coefficients (the admissible contracts in BVW (2016)) are time-inconsistent – the investor would optimally want to change the coefficients as the time goes

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by. This makes the problem difficult.

Our asset pricing results are similar to those of the OU case in BVW (2014): the stocks in large supply have high risk premia, and the stocks in low supply have low risk premia, and this effect is stronger as agency friction increases. However, by using the contract that provides optimal risk-taking incentives, the sensitivity of the price distortion to agency frictions is of an order of magnitude smaller compared to the price distortion in BVW (2014). In other words, by including the risk-incentive terms in the compensation, the investor mitigates somewhat the effect of agency frictions in equilibrium.

Other than BVW (2014) and the current paper, the existing literature either looks at the case of a fixed contract and then finds asset prices in equilibrium, or the case of fixed asset prices and then finds the optimal contract. In the first strand of the literature with fixed contracts, none of the papers, other than the current one, allows for quadratic variation and co-variation components in the contract. That literature includes the following papers (a more thorough literature review can be found in BVW 2014): Brennan (1993) considers a static model with preferences based on a benchmark, resulting in a two-factor equilibrium model; Basak and Pavlova (2013) consider a similar set-up, but in a dynamic model; Cuoco and Kaniel (2011) have a dynamic setting with two risky assets, and the contract is a piece-wise affine function of the portfolio return and the return relative to a benchmark; Malamud and Petrov (2014) consider two types of managers, less and more informed.

The second strand of the literature with fixed asset prices includes the following papers: Ou-Yang (2003) has a dynamic model in which the portfolio value is only observable at the terminal time, and in which there is no moral hazard due to shirking, so that the optimal contract does not have quadratic variation/covariation components; Cadenillas et al. (2007) extend some of Ou-Yang (2003) results to non-CARA utility functions, still with no moral hazard; Lioui and Poncet (2013) assume that the agent has enough bargaining power to require that the contract be linear in the output and in a benchmark factor; Leung (2014) studies a model with a single risky asset, in which moral hazard arises because there is an exogenous factor multiplying the volatility choice of the agent, and that factor is not observed by the principal; CPT (2016ab) find the optimal contract when the primary source of moral hazard is not due to shirking, but to the volatility vector being unobserved and the agent's cost of modifying it. Their model has finite horizon T and the agent is paid with a lump-sum contract payment at T only, unlike the present paper in which the payments are continuous over an infinite horizon.

The rest of the paper is organized as follows: Section 2 sets up the model and the optimization problems, Section 3 describes the main results, Section 4 extends the result to the case when Agent can invest privately in the index, Section 5 concludes, and Section 6 provides the proofs.

Some notational conventions. Let $(\Omega, \mathbf{F} = \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P})$ denote a filtrated probability space, whose filtration $\mathbf{F}$ is the augmented filtration generated by independent Brownian motions $B^p, (B^e)_{i=1,\dots,N}$,

and satisfies the usual conditions of completeness and right-continuity. For a **F**-adapted process X, $\mathbf{F}^X$ denotes the filtration generated by X and satisfies the usual conditions.

2 Model

2.1 Assets

The market consists of a riskless asset with an exogenous constant risk-free rate r, and N risky assets whose prices $(S_{it})_{i=1,\dots,N}$ will be determined in equilibrium. We work with the following model considered by Buffa et al. (2014), henceforth BVW (2014). Assume that the dividend process of asset $i = 1, \dots, N$ is given by

$$D_{it} = a_i p_t + e_{it}, \tag{2.1}$$

where p and e_i follow Ornstein-Uhlenbeck processes

$$\begin{aligned} dp_t &= \kappa^p(\bar{p} - p_t)dt + \sigma_p dB_t^p, \\ de_{it} &= \kappa_i^e(\bar{e}_i - e_{it})dt + \sigma_{ei}dB_{it}^e. \end{aligned} \tag{2.2}$$

Here, B^p and $(B_i^e)_{i=1,\dots,N}$ are independent Brownian motions, and model coefficients $a_i, \bar{p}, \bar{e}_i, \kappa^p, \kappa_i^e$, σ_p, σ_{ei}, for $i = 1, \dots, N$, are all positive constants. The filtration $\mathbf{F}^{B^p, B^e}$, denoted by $\mathbf{F}$, represents the full information in the model. We introduce the following vector and matrix notation for future use:

$$e = \text{diag}\{e_1, \dots, e_N\}, \quad \bar{e} = \text{diag}\{\bar{e}_1, \dots, \bar{e}_N\}, \quad \sigma_e = \text{diag}\{\sigma_{e1}, \dots, \sigma_{eN}\},$$
$$D = (D_1, \dots, D_N)', \quad S = (S_1, \dots, S_N)', \quad \kappa^e = (\kappa_1^e, \dots, \kappa_N^e)', \quad B^e = (B_1^e, \dots, B_N^e)'.$$

The vector of assets' return per share in excess of the riskless rate follows

$$dR_t = D_t dt + dS_t - rS_t dt. \tag{2.3}$$

The excess return of the market portfolio, or index, is given by

$$I_t = \eta' R_t, \tag{2.4}$$

where $\eta = (\eta_1, \dots, \eta_N)'$ is a constant vector, with η_i equal to the number of shares of asset i in the market. However, we assume that not all the shares of assets are available for trade. A constant vector $\theta = (\theta_1, \dots, \theta_N)'$, with entries equal to the number of shares available to trade is called the *residual supply*. The difference $\eta_i - \theta_i$ equals the number of shares of asset i held by buy-and-hold investors who do not trade. We assume that each component of θ is strictly positive.

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2.2 Agent and Principal

In addition to buy-and-hold investors, there are two market participants in the model: Agent (portfolio manager) and Principal (investor). They can be considered as representatives of identical agents and principals. Both Agent and Principal are price-takers, that is, they take prices as given, without taking into account the feedback effects in equilibrium.

Principal can hire Agent to manage a portfolio of assets on Principal's behalf. Agent, if hired by Principal, receives compensation (fee) paid by Principal, manages a portfolio of assets, and he can also undertake a "shirking" action that has a detrimental effect on the portfolio, but it provides Agent with a private benefit.

In the benchmark model, Agent can only invest in the riskless asset in his private account, and he can also consume from it. (An extension where Agent is allowed to trade privately in the index is discussed in Section 4.) Thus, Agent is exposed to the risky assets only via the compensation paid by Principal. Agent's wealth process is given by

$$d\bar{W}_t = r\bar{W}_t dt + (bm_t - \bar{c}_t)dt + dF_t, \tag{2.5}$$

where

- $\bar{c}_t$ is Agent's consumption rate;
- bm_t is the private benefit from his rate m_t of shirking with the benefit rate $b \in [0, 1]$;
- F_t is the cumulative compensation paid by Principal.

Principal can trade in the index, but not in the individual risky assets. The only way she can access individual risky assets is by hiring Agent. Principal's wealth process follows

$$dW_t = rW_t dt + dG_t + y_t dI_t - c_t dt - dF_t, \tag{2.6}$$

where:

- $G_t = \int_0^t Y_s' dR_s - m_s ds$ is the *reported* cumulative fund return process, where Y is the vector of the number of shares of the risky assets held by Agent in the managed portfolio;
- y_t is the number of shares of the index held by Principal;
- c_t is Principal's consumption rate.

Agent's rate m_t of shirking action m_t is assumed to be nonnegative. It reduces Principal's wealth; in addition to shirking, it can also be interpreted as diverting money from the portfolio for expenses that do not contribute to the performance of the fund. More generally, it may be thought of as a measure of (lack of) Agent's efficiency when running the portfolio; see DeMarzo and Sannikov (2006).

Agent maximizes utility over intertemporal consumption:

$$\bar{V} = \max_{\Xi \text{ admissible}} \mathbb{E}\left[\int_0^\infty e^{-\bar{\delta}t} u_A(\bar{c}_t)dt\right],$$

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where u_A is exponential utility with constant absolute risk aversion $\bar{\rho}$, i.e., $u_A(c) = -\frac{1}{\bar{\rho}}e^{-\bar{\rho}c}$, and $\bar{\delta} > 0$ is Agent's discounting rate. Given Agent's utility function u_A, we can assume, without loss of generality, that the initial wealth of Agent is zero, i.e., $\bar{W}_0 = 0$.

Given Principal's strategy $\Theta = (c, F, y)$, Agent's strategy $\Xi = (\bar{c}, m, Y)$ is *admissible* if the following conditions are satisfied

- $(\bar{c}, m, Y)$ is adapted to **F**;
- Y is predictable, $\int_0^t |Y_s|^2 ds < \infty$ for all $t > 0$;
- $m \geq 0$;
- $\bar{c}$ is financed by wealth process $\bar{W}$ satisfying (2.5).

If Agent is not employed by Principal, he chooses his private portfolio Y^u and consumption rate $\bar{c}^u$ to maximize his utility over consumption

$$\bar{V}^u = \max_{(Y^u, \bar{c}^u) \text{ admissible}} \mathbb{E}\left[\int_0^\infty e^{-\bar{\delta}t} u_A(\bar{c}_t^u) dt\right],$$

subject to the budget constraint

$$d\bar{W}_t^u = r\bar{W}_t^u dt + Y_t^u dR_t - \bar{c}_t^u dt. \tag{2.7}$$

Agent's private investment and consumption strategy $(Y^u, \bar{c}^u)$ is *admissible* if

- $(Y^u, \bar{c}^u)$ is adapted to **F**;
- Y^u is predictable, $\int_0^t |Y_s^u|^2 ds < \infty$ for all $t > 0$;
- $\bar{c}^u$ is financed by wealth process $\bar{W}^u$ satisfying (2.7).
- The following transversality condition is satisfied:

$$\lim_{T \to \infty} \lim_{n \to \infty} \mathbb{E}\left[e^{-\bar{\delta}(T \wedge \tau_n)} e^{-r\bar{\rho}\bar{W}_{T \wedge \tau_n}^u}\right] = 0,$$

for any sequence of stopping time $\{\tau_n\}_n$ with $\lim_n \tau_n = \infty$.

Agent takes the contract offered by Principal if and only if the following *participation constraint* is satisfied:

$$\bar{V} \geq \bar{V}^u. \tag{2.8}$$

When this inequality is an equality, Agent is indifferent with respect to taking the contract or not. In this case, as is standard in contract theory, we assume that Agent chooses to work for Principal.

Principal maximizes utility over intertemporal consumption:

$$V = \max_{\Theta \text{ admissible}} \mathbb{E}\left[\int_0^\infty e^{-\delta t} u_P(c_t) dt\right],$$

where u_P is an exponential utility with constant absolute risk aversion ρ, i.e., $u_P(c) = -\frac{1}{\rho}e^{-\rho c}$, and $\delta > 0$ is Principal's discounting rate. Principal's strategy $\Theta = (c, F, y)$ is *admissible* if

- Agent's optimization problem admits at least one admissible optimal strategy $\Xi^* = (\bar{c}^*, m^*, Y^*)$;

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- (c, F, y) is adapted to $\mathbf{F}^{G^*, I}$, where $G_t^* = \int_0^t (Y_s^*)' dR_s - m_s^* ds$ is the reported cumulative fund return when Agent employs his optimal strategy Ξ^*.

- y is predictable, $\int_0^t y_s^2 ds < \infty$ for all $t \geq 0$;

- The consumption stream c is financed by the wealth process W satisfying

$$dW_t = rW_t dt + dG_t^* + y_t dI_t - c_t dt - dF_t.$$

If Principal does not hire Agent, she chooses investment y^u in the index and consumption rate c^u to maximize her utility over consumption

$$V^u = \max_{(y^u, c^u) \text{ admissible}} \mathbb{E}\left[\int_0^\infty e^{-\delta t} u_p(c_t^u) dt \right],$$

subject to the budget constraint

$$dW_t^u = rW_t^u dt + y_t^u dI_t - c_t^u dt. \tag{2.9}$$

Principal's private investment and consumption strategy (y^u, c^u) is *admissible* if

- (y^u, c^u) is adapted to $\mathbf{F}^I$;
- y^u is predictable, $\int_0^t |y_s^u|^2 ds < \infty$ for all $t > 0$;
- c^u is financed by wealth process W^u satisfying (2.9).
- The following transversality condition is satisfied:

$$\lim_{T \to \infty} \lim_{n \to \infty} \mathbb{E}\left[e^{-\delta(T \wedge \tau_n)} e^{-r\rho W_{T \wedge \tau_n}^u} \right] = 0,$$

for any sequence of stopping time $\{\tau_n\}_n$ with $\lim_n \tau_n = \infty$.

Principal hires Agent if and only if

$$V \geq V^u. \tag{2.10}$$

When the inequality above is an equality, Principal is indifferent to hiring Agent or not. In this case, we assume that Principal chooses to hire Agent.

2.3 Equilibrium

We will look for equilibria in which Principal hires Agent. The notion of equilibrium is similar to BVW (2014). The only difference is that the class of the contracts which Principal is allowed to optimize over is incorporated in our notion of equilibrium.

Definition 2.1 *A price process S, a contract F in a class of contracts $\mathscr{F}$, and an index investment y form an equilibrium if:*

(i) Given S, $(F, \mathscr{F})$ and y, Agent takes the contract, and $Y = \theta - y\eta$ solves Agent's optimization problem.

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(ii) Given S, Principal hires Agent, contract F is optimal for Principal in the class $\mathcal{F}$ and y is her optimal index investment strategy.

We will look for the equilibrium in which the price of asset i is of the form

$$S_{it} = a_{0i} + a_{pi}p_t + a_{ei}e_{it}, \tag{2.11}$$

where (a_{0i}, a_{pi}, a_{ei}) are constants that will be determined in equilibrium. The form of (2.11), combined with (2.1), (2.2), and (2.3), imply that the excess return for asset i follows

$$\begin{aligned}
dR_{it} &= \left[(a_i - a_{pi}(r + \kappa^p))p_t + (1 - a_{ei}(r + \kappa_i^e))e_{it} + \kappa^p a_{pi}\bar{p} + \kappa_i^e a_{ei}\bar{e}_i - ra_{0i}\right]dt \\
&\quad + a_{pi}\sigma_p dB_t^p + a_{ei}\sigma_i dB_{it}^e \\
&=: [A_{1i}p_t + A_{2i}e_{it} + A_{3i}]dt + a_{pi}\sigma_p dB_t^p + a_{ei}\sigma_{ei}dB_{it}^e. \tag{2.12}
\end{aligned}$$

Denote also

$$\gamma = (a_{p1}, \ldots, a_{pN})'\sigma_p, \quad \sigma = \operatorname{diag}\{a_{e1}, \ldots, a_{eN}\}\sigma_e, \quad A_\ell = (A_{\ell 1}, \ldots, A_{\ell N})', \quad \ell = 1, 2, 3,$$

$$\mu_t - r = p_t A_1 + e_t A_2 + A_3, \quad \text{and} \quad \Sigma_R = \gamma\gamma' + \sigma^2.$$

Then, the vector of asset returns follows

$$dR_t = (\mu_t - r)dt + \gamma dB_t^p + \sigma dB_t^e, \tag{2.13}$$

with (instantaneous) covariance matrix Σ_R.

3 Optimal strategies and equilibrium

3.1 Family of viable contracts

When defining the family of contracts that Principal can choose from, we follow the approach of Cvitanić, Possamai and Touzi (2016ab), henceforth CPT (2016ab). In their framework, they show that the approach represents no loss of generality (under technical conditions), that is, that Principal attains maximal utility when optimizing over the family they define. While we have not proved that result in our framework, we conjecture that a similar result is still true under reasonable technical conditions. A verification of this conjecture requires a substantial new development of the theory of 2BSDE on infinite horizon, which is outside of the scope of this paper.

The approach consists of defining the family of viable contracts for which Agent's problem satisfies the dynamic programming principle. Let us first motivate the definition of a viable contract. For $t \geq 0$ and a given Agent's admissible strategy $\Xi = (\bar{c}, Y, m)$, consider the following class of admissible strategies

$$\Xi^t = \{\hat{\Xi} \text{ admissible} \mid \hat{\Xi}_s = \Xi_s, s \in [0,t]\}.$$

Define Agent's *continuation value process* $\bar{\mathcal{V}}(\Xi)$ as

$$\bar{\mathcal{V}}_t(\Xi) = \text{ess sup}_{\Xi^t}\, \mathbb{E}_t\left[\int_t^\infty e^{-\bar{\delta}(s-t)}u_A(\bar{c}_s)ds\right], \quad t \geq 0.$$

That is, $\bar{\mathcal{V}}_t(\Xi)$ is Agent's optimal value at time t if he employs the strategy Ξ before time t and acts optimally from time t onward. The continuation value process is expected to satisfy the *martingale principle*, which can be viewed as the dynamic programming principle in non-Markovian settings; i.e., process $\tilde{\mathcal{V}}(\Xi)$, defined as

$$\tilde{\mathcal{V}}_t(\Xi) = e^{-\bar{\delta}t}\bar{\mathcal{V}}_t(\Xi) + \int_0^t e^{-\bar{\delta}s}u_A(\bar{c}_s)ds,$$

is a supermartingale for arbitrary admissible strategy Ξ, and is a martingale for the optimal strategy Ξ^*. The following result provides two properties of the continuation value in our setting.

Lemma 3.1 *For any $t \geq 0$ and admissible Ξ,*

  (i) $\partial_{\bar{W}_t}\bar{\mathcal{V}}_t(\Xi) = -r\bar{\rho}\,\bar{\mathcal{V}}_t(\Xi)$;

  (ii) $\lim_{t\to\infty}\mathbb{E}\left[e^{-\bar{\delta}t}\bar{\mathcal{V}}_t(\Xi)\right] = 0.$

These properties of the continuation value motivates us to introduce the following family of Principal's strategies. In this definition, we assume that Σ_R is invertible. First, we introduce the process P via

$$dP_t = (bm_t - \bar{c}_t)dt, \quad P_0 = 0,$$

which records the impact of Agent's private action on his wealth. Next, for real numbers $X > 0, Z \geq b, U, \Gamma^G < 0, \Gamma^I, \Gamma^{GI}$ define the Hamiltonian H by

$$H(X, Z, U, \Gamma^G, \Gamma^I, \Gamma^{GI}) = \sup_{(\bar{c}, m \geq 0, Y)}\left\{u_A(\bar{c}) + X\left[bm - \bar{c} - Zm + ZY'(\mu - r) + U\eta'(\mu - r)\right.\right.$$
$$\left.\left. + \tfrac{1}{2}\Gamma^G Y'\Sigma_R Y + \tfrac{1}{2}\Gamma^I \eta'\Sigma_T \eta + \Gamma^{GI}Y'\Sigma_R \eta\right]\right\}.$$
$$(3.1)$$

Let us note that if the model was Markovian in $(\bar{W}, G, I)$, X would be the derivative of the value function $\bar{\mathcal{V}}(\bar{W}, G, I)$ with respect to $\bar{W}$, and similarly $XZ, XU, X\Gamma^G, X\Gamma^I, X\Gamma^{GI}$ would be the first and second derivatives with respect to $\bar{W}, G$ and I.

Definition 3.2 *Principal's admissible strategy $\Theta = (c, F, y)$ is* viable *if there exist*

  • *a constant $\bar{V}_0$ and*

  • *a class of Agent's admissible strategies $\Xi(\Theta)$,*

such that

(a) for any Agent's strategy $\Xi \in \Xi(\Theta)$, there exist $\mathbf{F}^{G,I}$ adapted processes $Z, U, \Gamma^G, \Gamma^I, \Gamma^{GI}$, satisfying $\int_0^t Z_s^2 ds < \infty$, $\int_0^t U_s^2 ds < \infty$, for all $t > 0$, and $Z \geq b$, $\Gamma^G < 0$ such that the process $\bar{V}(\Xi)$ defined by

$$d\bar{V}_t(\Xi) = X_t \left[dP_t + Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma_t^G d\langle G \rangle_t + \tfrac{1}{2}\Gamma_t^I \langle I \rangle_t + \Gamma_t^{GI} d\langle G, I \rangle_t \right]$$
$$+ \bar{\delta}\bar{V}_t(\Xi) dt - H(Z_t, U_t, \Gamma_t^G, \Gamma_t^I, \Gamma_t^{GI}) dt, \quad \bar{V}_0(\Xi) = \bar{V}_0, \tag{3.2}$$

where $X_t = -r\bar{\rho}\bar{V}_t(\Xi)$, satisfies the transversality condition

$$\lim_{T \to \infty} \lim_{n \to \infty} \mathbb{E}\left[e^{-\bar{\delta}T \wedge \tau_n} \bar{V}_{T \wedge \tau_n}(\Xi) \right] = 0, \tag{3.3}$$

for any sequence of stopping times $\{\tau_n\}_n$ with $\lim_n \tau_n = \infty$.

(b) the class $\Xi(\Theta)$ contains a strategy $\Xi^* = (\bar{c}^*, m^*, Y^*)$ that maximizes the Hamiltonian, that is, the strategy with

$$\bar{c}^* = (u_A')^{-1}(-r\bar{\rho}\bar{V}(\Xi^*)), \quad m^* = 0, \quad Y^* = -\frac{Z}{\Gamma^G}\Sigma_R^{-1}(\mu - r) - \frac{\Gamma^{GI}}{\Gamma^G}\eta; \tag{3.4}$$

(c) Denoting the reported portfolio value and the contract value by G^* and F^*, respectively, when Agent employs strategy Ξ^*, then, Principal's wealth process, following the dynamics

$$dW_t = rW_t dt + dG_t^* + y dI_t - c_t dt - dF_t^*, \tag{3.5}$$

satisfies the transversality condition

$$\lim_{T \to \infty} \lim_{n \to \infty} \mathbb{E}\left[e^{-r\rho W_{T \wedge \tau_n}} \right] = 0, \tag{3.6}$$

for any sequence of stopping times $\{\tau_n\}_n$ with $\lim_n \tau_n = \infty$.

The next lemma will show that when Principal employs a viable strategy, then strategy Ξ^* in (3.4) is Agent's optimal strategy and $\bar{V}(\Xi)$ is Agent's continuation value process $\mathscr{V}(\Xi)$. The dynamics (3.2) and the definition of H in (3.1) are motivated by the martingale principle, in particular, (3.2) and (3.1) ensure that $\tilde{V}_t(\Xi) = e^{-\bar{\delta}t}\bar{V}_t(\Xi) + \int_0^t e^{-\bar{\delta}s} u_A(\bar{c}_s) ds$ is a supermartingale for an arbitrary admissible strategy Ξ, and is a martingale for strategy Ξ^*. Moreover (3.2) gives a stochastic representation for Agent's continuation value process, with sensitivities with respect to $P, G, I, \langle G \rangle, \langle I \rangle, \langle G, I \rangle$ given by processes $X, XZ, XU, \tfrac{1}{2}X\Gamma^G, \tfrac{1}{2}X\Gamma^I$, and $X\Gamma^{GI}$, respectively. Since $\Gamma^G, \Gamma^I, \Gamma^{GI}$ can be arbitrary $\mathbf{F}^{G,I}$-adapted processes (with $\Gamma^G < 0$), the viable strategy allows all possible sensitivities with respect to quadratic variation and covariations of G and I. Note also that the sensitivity of Agent's continuation value with respect to P is the same as the sensitivity with respect to $\bar{W}$. This is why, taking Lemma 3.1 into account, X is set to be equal to $-r\bar{\rho}\bar{V}(\Xi)$.

The reason we require $Z \geq b$ is that, when $Z_t < b$ for t, Hamiltonian H is maximized for $m = \infty$. This would lead to Principal's wealth being equal to $-\infty$, hence not optimal for Principal. When

$Z = b$, all nonnegative values of m maximize the Hamiltonian, and Agent is indifferent which m to choose. In this case, we follow the usual convention in contract theory and assume that Agent will choose the best value for Principal, i.e., $m = 0$.

Lemma 3.3 *Consider any Principal's viable strategy $\Theta = (c, F, y)$. Assume that Σ_R is invertible. Then, the strategy $\Xi^* = (\bar{c}^*, m^*, Y^*)$ in (3.4) is Agent's optimal strategy in the class $\Xi(\Theta)$, and $\bar{V}_0$ is Agent's optimal value at time 0. Moreover, $\bar{V}(\Xi)$ is equal to Agent's continuation value process $\bar{\mathscr{V}}(\Xi)$.*

In CPT (2016ab) the compensation is paid only at the terminal time T. Therefore, the form of a viable contract payment F_T is recognized from the fact that $\bar{V}_T(\Xi) = F_T$. In the present case, the compensation is paid continuously and it does not show up explicitly in (3.2). The following result provides the form of the contract in a viable strategy.

Lemma 3.4 *Contract F in any viable strategy satisfies*

$$dF_t = Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma_t^G d\langle G\rangle_t + \tfrac{1}{2}\Gamma_t^I d\langle I\rangle_t + \Gamma_t^{GI} d\langle G,I\rangle_t + \tfrac{1}{2}r\bar{\rho}\, d\langle Z\cdot G + U\cdot I\rangle_t \\ - \left(\tfrac{\bar{\delta}}{r\bar{\rho}} + \bar{H}_t\right)dt, \tag{3.7}$$

where $\langle G\rangle$ denotes the quadratic variation of G, $Z\cdot G = \int_0^{\cdot} Z_s dG_s$, and

$$\bar{H}_t = \tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) - \tfrac{1}{\bar{\rho}} + (Z_t Y_t^* + U_t\eta)'(\mu_t - r) + \tfrac{1}{2}\Gamma_t^G (Y_t^*)'\Sigma_R Y_t^* + \tfrac{1}{2}\Gamma_t^I \eta'\Sigma_R\eta + \Gamma_t^{GI}(Y_t^*)'\Sigma_R\eta. \tag{3.8}$$

In particular, when $\mu - r$ is a constant vector, F is adapted to $\mathbf{F}^{G,I}$.

Remark 3.5 *The lemma shows that a viable contract is linear, in the integration sense, with respect to G, I, their quadratic variation and covariations, and the quadratic variation $\langle Z\cdot G + U\cdot I\rangle$ of Agent's wealth $\bar{W}$. In particular, the linear contracts considered in BVW (2014) of the form, for some constants ϕ, χ, and ψ,*

$$dF_t^{BVW} = \phi dG_t - \chi dI_t + \psi dt \tag{3.9}$$

are viable. Indeed, we can choose Γ^G, Γ^I and Γ^{GI} so that all the quadratic variation/covariation terms sum up to zero.

We could have simply started by requiring that a viable contract is of the form (3.7), for an arbitrary adapted process H_t (and not including the term $\langle Z\cdot G + U\cdot I\rangle$). However, then, it wouldn't have been clear how to solve Agent's problem for arbitrary adapted processes Z, U, Γ^G, Γ^I, and Γ^{GI} satisfying the above conditions, and, more importantly, our approach shows why the contracts of the form (3.7) are as general as can be expected if Agent's problem can be solved by the dynamic programming principle.

3.2 Main results

Let us introduce some notation before stating the main results:

- The instantaneous variance of the index portfolio:

$$Var^\eta = \eta' \Sigma_R \eta.$$

- The instantaneous variance of the fund portfolio for the fund that invests Y in risky assets:

$$Var^Y = Y' \Sigma_R Y.$$

-The instantaneous covariance between the fund portfolio and the index portfolio:

$$Covar^{Y,\eta} = \eta' \Sigma_R Y.$$

- The CAPM beta of the fund portfolio:

$$\beta^Y = \frac{Covar^{Y,\eta}}{Var^\eta}.$$

3.2.1 Optimal strategies

Given asset prices, not necessarily in asset pricing equilibrium, we first state the results on optimal strategies.

Theorem 3.6 *Consider a financial market in which the vector of asset returns (per share) has a constant drift vector μ and constant covariance matrix Σ_R such that Σ_R is invertible and $\eta' \Sigma_R \eta > 0$. Assume that Principal can attain a higher value than V^u by hiring Agent, and Agent can attain a higher value than $\bar{V}^u$ by working for Principal. Then, one optimal strategy for Principal is not to invest in the index, and*

(a) The optimal contract in the viable class is given by

$$dF_t = Cdt + \frac{\rho}{\rho+\bar{\rho}}dG_t + \xi(dG_t - \beta^{Y^*}dI_t) + \frac{r}{2}\zeta d\langle G - \beta^{Y^*}I\rangle_t, \tag{3.10}$$

where G is the reported portfolio return process,

$$
\begin{aligned}
\xi &= (b - \tfrac{\rho}{\rho+\bar{\rho}})_+, \\
\zeta &= (\rho + \bar{\rho})Z(1-Z)(b - \tfrac{\rho}{\rho+\bar{\rho}})_+, \\
C &= \tfrac{1}{2r}\tfrac{1}{\bar{\rho}}(\mu - r)'\Sigma_R^{-1}(\mu - r) - (ZY^* + U\eta)'(\mu - r) \\
&\quad - \tfrac{r}{2}\zeta(Y^* - \beta^{Y^*}\eta)'\Sigma_R(Y^* - \beta^{Y^*}\eta) + \tfrac{r}{2}\bar{\rho}(ZY^* + U\eta)'\Sigma_R(ZY^* + U\eta), \\
Z &= \max\{b, \tfrac{\rho}{\rho+\bar{\rho}}\} = \tfrac{\rho}{\rho+\bar{\rho}} + (b - \tfrac{\rho}{\rho+\bar{\rho}})_+, \\
U &= -(b - \tfrac{\rho}{\rho+\bar{\rho}})_+\beta^{Y^*}.
\end{aligned}
\tag{3.11}
$$

(b) Agent's vector of optimal holdings is given by

$$Y^* = \frac{1}{r}\frac{1}{\mathscr{C}_b}\Sigma_R^{-1}(\mu - r) + \frac{1}{r}\Big(\frac{\rho + \bar{\rho}}{\rho\bar{\rho}}\frac{\mathscr{D}_b}{\mathscr{C}_b}\Big)\frac{\eta'(\mu - r)}{Var\eta}\eta, \tag{3.12}$$

where

$$\mathscr{D}_b = (\rho + \bar{\rho})\big(b - \tfrac{\rho}{\rho + \bar{\rho}}\big)_+^2, \tag{3.13}$$
$$\mathscr{C}_b = \tfrac{\rho\bar{\rho}}{\rho + \bar{\rho}} + \mathscr{D}_b.$$

(c) Principal's value process V_t is of the form

$$V_t = V(W_t) = Ke^{-r\rho W_t}, \tag{3.14}$$

for an appropriate constant K.

(d) Agent's value process satisfies the linear SDE

$$d\bar{V}_t = \bar{V}_t\big[-r\bar{\rho}(ZY + U\eta)'(\gamma dB_t^p + \sigma dB_t^\sigma) + (\bar{\delta} - r)dt\big]. \tag{3.15}$$

3.3   Contract properties

1. **First best.** If $b \le \frac{\rho}{\rho + \bar{\rho}}$, Principal can attain the first best utility, the one she would get if she was the one choosing portfolio holdings Y rather than Agent choosing them. In this case $\xi = \zeta = 0$ in (3.10), and Agent receives the fraction $\frac{\rho}{\rho + \bar{\rho}}$ of the reported portfolio return. That is, the optimal contract does not have quadratic variation term and is equal to

$$dF_t = \tfrac{\rho}{\rho + \bar{\rho}}dG_t.$$

   The fraction $\frac{\rho}{\rho + \bar{\rho}}$ is the classical risk-sharing fraction of the wealth between two agents with CARA utilities. Moreover, in this case $\mathscr{D}_b = 0$, and there is no agency friction.

2. **Second best.** As mentioned above, in the optimal contract (3.10), the term $\frac{\rho}{\rho + \bar{\rho}}dG_t$ is the risk-sharing term that is the only incentive part of the contract in the first best case of no agency friction. The term $\xi(dG_t - \beta^{Y^*}dI_t)$ benchmarks the reported portfolio return against the portfolio that invests β^{Y^*} in the index. We can think of $\beta^{Y^*}I$ as the approximation of the portfolio with strategy Y^* which is optimal (in L_2 sense) among the approximations of the form cI_t for some constant c. We call this portfolio the *optimal benchmark portfolio*. Thus, $\xi(dG_t - \beta^{Y^*}dI_t)$ rewards Agent when the portfolio return is above the return of the optimal benchmark portfolio, and penalizes Agent when the portfolio return is below the return of the optimal benchmark portfolio.

   When $b \le \frac{\rho}{\rho + \bar{\rho}}$, the quadratic variation and covariation parts of the contract are zero. However, when $b > \frac{\rho}{\rho + \bar{\rho}}$, there is a new quadratic variation term compared to BVW (2014). This

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new term provides additional incentives for aligning Agent's risk taking with Principal's objectives by rewarding the quadratic variation of the deviation $G - \beta^{Y^*}I$ from the optimal benchmark portfolio. This quadratic variation can be viewed as the "tracking gap" between Agent's portfolio and the optimal benchmark portfolio. Note that when $b > \frac{\rho}{\rho + \bar{\rho}}$ the sensitivity with respect to $\langle G - \beta^\theta I \rangle_t$ is $\frac{r}{2}\zeta$, which is positive, thus rewarding Agent for deviating from the optimal benchmark portfolio. Thus, the quadratic variation term rewards Agent for taking the specific risk of individual stocks, and not only the systematic risk of the index.

When agency friction b increases, ξ increases, so as to make Agent to not employ the shirking action. As a result, the portfolio is benchmarked more heavily to the optimal benchmark portfolio. Dependence of ζ on the agency friction is demonstrated in Figure 1. When agency friction is small, ζ increases with respect to agency friction, so that Agent is increasingly awarded by taking specific risks. However when agency friction is large, the benefits of taking that specific risk is lower, therefore ζ decreases with respect to agency friction, so that Agent is incentivized not to take as much specific risk.
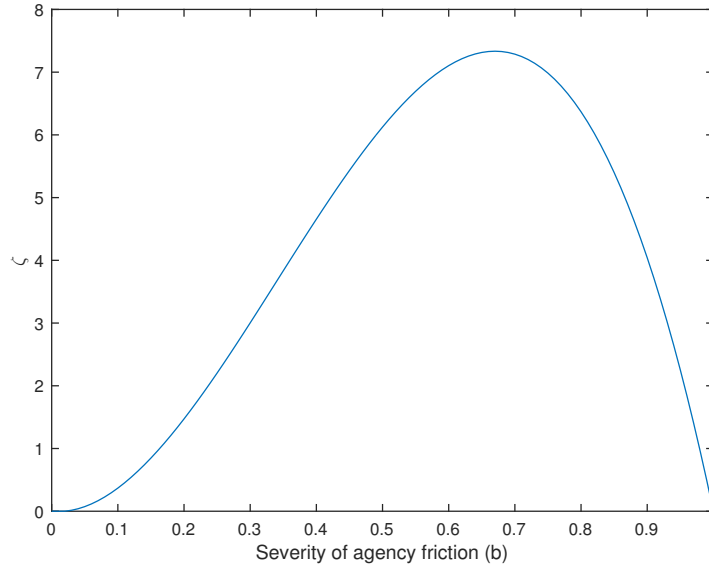

Figure 1: Sensitivity to quadratic variation

Finally, the quadratic variation term depends on the interest rate, but the profit sharing and benchmarking terms do not.

3. **Optimal fund holdings.** Note that $\Sigma_R^{-1}(\mu - r)$ is the vector of risk premia of the individual risky assets, and $\frac{\eta'(\mu - r)}{Var\eta}$ is the risk premium of the index. Therefore, item (c) in Theorem 3.9 shows that Agent's optimal holding in asset i is a linear combination of the risk premium of asset i and the portion of the risk premium of the index corresponding to asset i. Moreover,

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when agency friction increases, the weight to the individual asset risk premium decreases, while the weight to the index increases.

3.3.1 Equilibrium prices

We will need the following assumption for the equilibrium result.

Assumption 3.7

(i) θ *and* η *are not linearly dependent.*

(ii) *Denote* $a_p = (a_{p1}, \ldots, a_{pN})'$ *and* $a_e = \mathrm{diag}\{a_{e1}, \ldots, a_{eN}\}$. *For the values*

$$a_{pi} = \frac{a_i}{r + \kappa^p} \quad a_{ei} = \frac{1}{r + \kappa_i^e}, \quad i = 1, \ldots, N, \tag{3.16}$$

the matrix $\Sigma_R = a_p \sigma_p^2 a_p' + a_e' \sigma_e^2 a_e$ *is invertible.*

Remark 3.8 *Since Principal can invest in the index directly, if* $\theta = \alpha\eta$ *for some* $\alpha \in \mathbb{R}$, *then there exists an equilibrium in which Principal invests in* α *units of index directly without hiring Agent. Item (i) in the above assumption excludes this trivial case. Item (ii) ensures that the equilibrium we characterize is endogenously complete.*

The following is the main equilibrium result of the paper.

Theorem 3.9 *Suppose that Assumption 3.7 holds. Then, there exists an equilibrium in which Principal does not invest in the index directly, i.e.,* $y = 0$, *hence* $Y^* = \theta$, *in which asset prices are as in* (2.11) *and:*

(a) Vectors a_p *and* a_e *are given by* (3.16) *and vector* $a_0 = (a_{01}, \ldots, a_{0N})'$ *is given by, with* $\mathscr{D}_b$ *given in* (3.13),

$$a_0 = \frac{1}{r}\kappa^p \bar{p} a_p + \frac{1}{r}(\kappa^e)' \bar{e} a_e - \frac{\rho\bar{\rho}}{\rho + \bar{\rho}}\Sigma_R \theta - \mathscr{D}_b \Sigma_R (\theta - \beta^\theta \eta), \tag{3.17}$$

(b) The vector of asset excess returns is given by

$$\mu - r = r\frac{\rho\bar{\rho}}{\rho + \bar{\rho}}\Sigma_R \theta + r\mathscr{D}_b \Sigma_R (\theta - \beta^\theta \eta). \tag{3.18}$$

The index excess return is

$$\eta'(\mu - r) = r\frac{\rho\bar{\rho}}{\rho + \bar{\rho}}Covar^{\theta,\eta}. \tag{3.19}$$

The excess return of Agent's portfolio is

$$\theta'(\mu - r) = r\frac{\rho\bar{\rho}}{\rho + \bar{\rho}}Var^\theta + r\mathscr{D}_b\left(Var^\theta - \frac{(Covar^{\theta,\eta})^2}{Var^\eta}\right). \tag{3.20}$$

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(c) Principal offers optimally the contract that assigns value

$$\bar{V}_0 = \bar{V}_0^u = -\exp\left(1 - \tfrac{\bar{\delta}}{r} - \log(r\bar{\rho}) - \tfrac{1}{2r}(\mu - r)'\Sigma_R^{-1}(\mu - r)\right),$$

that is the minimal value Agent would accept. With this choice, Principal is always willing to offer the contract. Moreover, Principal's value process is given by $V(W_t) = Ke^{-r\rho W_t}$ *where*

$$K = -\exp\left(1 - \tfrac{\delta}{r} - \log(r\rho) + \tfrac{r}{2}\tfrac{\rho}{\bar{\rho}}(\mathscr{C}_b^2 - \bar{\rho}\mathscr{C}_b)Var^\theta + \tfrac{r}{2}\tfrac{\rho}{\bar{\rho}}(\mathscr{D}_b^2 - 2\mathscr{C}_b\mathscr{D}_b + \bar{\rho}\mathscr{D}_b)\tfrac{(Covar^{\theta,\eta})^2}{Var^\eta}\right).$$

$$(3.21)$$

3.4   Equilibrium properties

1. **Price and returns distortion.** Note that $\mathscr{D}_b$ increases with b. We see then, from (3.18), that the risk premium of asset i increases (resp. decreases) with b when $\theta_i/\eta_i > \beta^\theta$ (resp. $\theta_i/\eta_i < \beta^\theta$). That is, whether the risk premium goes up or down with agency frictions depends on how large is the fund's relative holding θ_i/η_i of asset i compared to the CAPM beta of the fund. Thus, the stocks in large supply have high risk premia, and the stocks in low supply have low risk premia, and this effect is stronger as agency friction increases.

   The price is distorted reversely. We see from (3.17) that the price of asset i decreases (resp. increases) with b when $\theta_i/\eta_i > \beta^\theta$ (resp. $\theta_i/\eta_i < \beta^\theta$). Therefore assets in large supply have lower prices and assets in low supply have higher price, and the effect is stronger as agency friction increases. This is the same qualitative behavior as in BVW (2014), Proposition 6.2. However, there is a quantitative difference. In BVW (2014), $\mathscr{D}_b$ is replaced by

   $$\mathscr{D}_b^{BVW} = \bar{\rho}(b - \tfrac{\rho}{\rho + \bar{\rho}})_+.$$

   Note that $\mathscr{D}_b < \mathscr{D}_b^{BVW}$ for any $b \in (0,1)$. Therefore, our price and returns distortions are less sensitive to agency friction than those in BVW (2014). Moreover, when agency friction is small, our sensitivities are of second order magnitude compared to the first order magnitude in BVW (2014). However, when $b = 1$, $\mathscr{D}_1$ and $\mathscr{D}_1^{BVW}$ are the same.

   Let us now take the same parameters as in BVW (2014): $\rho = 1, \bar{\rho} = 50, r = 4\%, \kappa^p = \kappa_i^e = 10\%, N = 6, \eta_i = 1, \theta_1 = \theta_2 = \theta_3 = 0.7, \theta_4 = \theta_5 = \theta_6 = 0.3, a_i = 1, \bar{p} = 0.65, \bar{e}_i = 0.4, \sigma_p = 1, \frac{\sigma_p^2}{\bar{p}} = \frac{\sigma_{ei}^2}{\bar{e}_i}$, for $i = 1, \ldots, 6$. Figure 2 compares distortion of excess return in our equilibrium with the one in BVW (2014).

2. **Portfolio returns.** As we see from (3.19), agency friction does not have impact on index excess return. This is because Principal can trade the index privately. However, (3.20) indicates that excess return of Agent's portfolio depends on agency friction. Since $Var^\theta > \frac{(Covar^{\theta,\eta})^2}{Var^\theta}$ by Cauchy-Schwarz inequality, the excess return of Agent's portfolio increases with agency friction. This means the increase in return of large supply assets dominates the
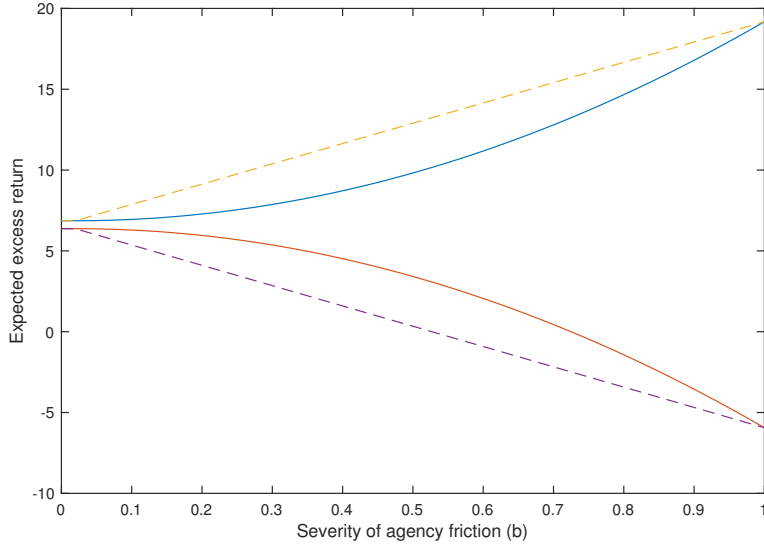
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Figure 2: Expected excess return of the two groups of assets. Assets with large supply are in the top half, assets with low supply are in the bottom half. The results of this paper are presented in solid lines, the results in BVW (2014) are presented in dashed lines.

decrease in return of low supply assets. Figure 3 demonstrates the excess return of Agent's portfolio in our equilibrium in comparison with BVW (2014).

Since the excess return of index does not change, the increase in Agent's portfolio return implies that there is a decrease in the return of the portfolio held by buy-and-hold investors. However, since $\mathscr{D}_b$ is lower in our paper, this means that buy-and-hold investors lose less compared to BVW (2014).

3. **Contract.** In equilibrium $Y^* = \theta$, so that from (3.19) we get

$$\beta^\theta = \frac{1}{r} \frac{\rho + \bar\rho}{\rho\bar\rho} \frac{\eta'(\mu - r)}{Var^\eta}$$

This is recognized as the optimal portfolio holding in the index in the case in which Agent and Principal can invest only in the index and they share the risk in the first best situation. We call this portfolio the *index-sharing portfolio*. Thus, $\xi(dG_t - \beta^\theta dI_t)$ rewards Agent when the portfolio return is above the return of the index-sharing portfolio, and penalizes Agent when the portfolio return is below the return of the index-sharing portfolio; the term $\zeta d\langle G - \beta^\theta I \rangle_t$ rewards Agent for taking specific risk of individual stocks, and not just the risk of the index-sharing portfolio.

4. **Principal's optimal value.** Given the excess return in BVW (2014)

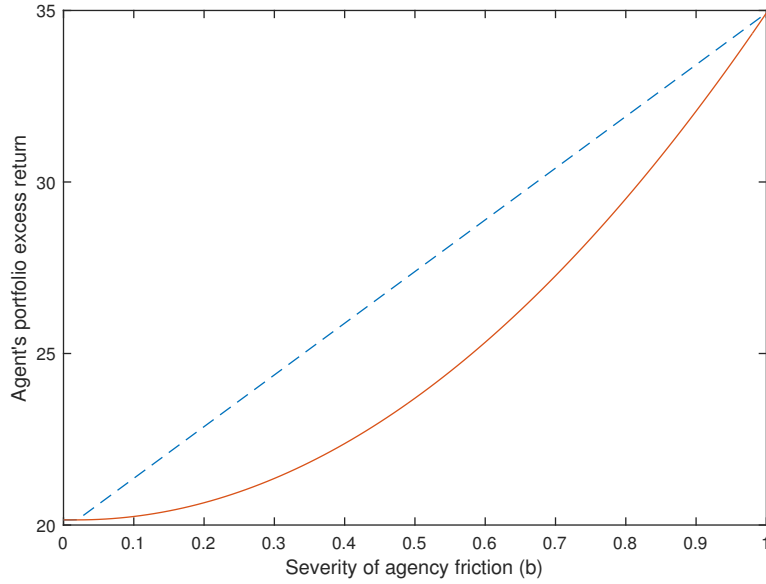$$\mu - r = r\bar\rho \Sigma_R (Z\theta + U\eta),$$

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Figure 3: Expected excess return of Agent's portfolio. The result of this paper is presented as a solid line, the result in BVW is presented in dashed lines.

and the same parameters as in BVW (2014) (see Figure 2), Figure 4 demonstrates that Principal's certainty equivalence could be improved substantially when the contract (3.10) is employed compared to (3.9). However, under this new contract, holding the residual demand to clear the market is no longer optimal for Agent. Therefore, the equilibrium in BVW (2014) fails to be an equilibrium when Principal is allowed to choose contracts from our viable class.

4 Extension: Agent can invest privately in the index

In this section, we extend the baseline model from Section 2 to the case in which Agent is allowed to invest privately in the index.

When Agent holds $\bar{y}_t$ shares of index at time t his wealth process $\bar{W}$ follows

$$d\bar{W}_t = (r\bar{W}_t + bm_t - \bar{c}_t)dt + \bar{y}_t dI_t + dF_t. \tag{4.1}$$

The admissibility of Agent's strategy $\Xi = (\bar{c}, m, Y, y)$ is defined similarly as in Section 2 with the additional requirement that $\bar{y}$ is predictable and satisfies $\int_0^t \bar{y}_s^2 ds < \infty$ for all $t > 0$. Principal's optimization problem is the same before, except, we assume that Principal can observe Agent's wealth process continuously. Equivalently, define the process P as

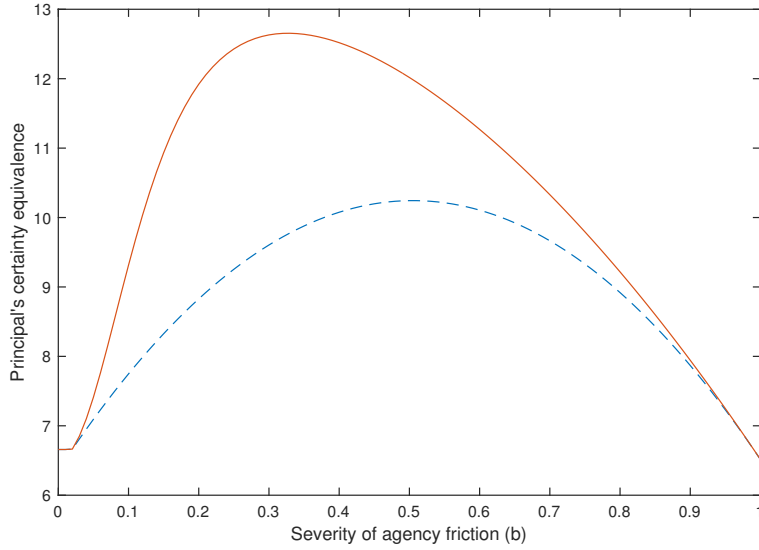$$dP_t = (bm_t - \bar{c}_t)dt + \bar{y}_t dI_t,$$

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Figure 4: Certainty equivalence of Principal when asset prices are as in BVW (2014). Solid line represents Principal's certainty equivalence when contract (3.10) is implemented, dashed line represents Principal's certainty equivalence when the contract in BVW (2014) is implemented.

which records the contribution to Agent's wealth through his private actions. Since Principal knows the contract F, she can observe P if and only if she can observe $\bar{W}$. We assume that this is the case, hence P is contractible. More precisely, we say that Principal's strategy $\Theta = (c, F, y)$ is admissible if it is adapted to $\mathbb{F}^{G,I,P}$. The notion of equilibrium in Definition (2.1) is modified accordingly, so that in item (i) Agent's optimal investment strategy Y and $\bar{y}$ satisfy $Y + \bar{y}\eta = \theta - y\eta$.

Then, for real numbers $X > 0, Z \geq b, U, \Gamma^G < 0, \Gamma^I, \Gamma^{GI}, \Gamma^P, \Gamma^P, \Gamma^{GP}$, such that $\Gamma^G \Gamma^P - (\Gamma^{GP})^2 > 0$, define the Hamiltonian H by

$$H = \sup_{(\bar{c}, m \geq 0, Y, \bar{y})} \left\{ u_A(\bar{c}) + X \left[bm - \bar{c} - Zm + \bar{y}\eta'(\mu - r) + ZY'(\mu - r) + U\eta'(\mu - r) \right. \right.$$
$$+ \tfrac{1}{2}\Gamma^G Y' \Sigma_R Y + \tfrac{1}{2}\Gamma^P \bar{y}^2 \eta' \Sigma_R \eta + \tfrac{1}{2}\Gamma^I \eta' \Sigma_R \eta$$
$$\left. \left. + \Gamma^{GI} Y' \Sigma_R \eta + \Gamma^{PI} \bar{y}\eta' \Sigma_R \eta + \Gamma^{GP} \bar{y} Y' \Sigma_R \eta \right] \right\}.$$

For invertible Σ_R, we define Principal's viable strategies as follows.

Definition 4.1 *Principal's strategy* $\Theta = (c, F, y)$ *is* viable, *if there exist*

- *a constant* $\bar{V}_0$;

- *a class of Agent's admissible strategies* $\Xi(\Theta)$;

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such that

(a) *for any Agent's strategy* $\Xi \in \Xi(\Theta)$, *there exist* $\mathbf{F}^{G,I,P}$ *adapted processes* $Z, U, \Gamma^G, \Gamma^P, \Gamma^I, \Gamma^{GI},$ Γ^{PI}, Γ^{GP}, *satisfying* $\int_0^t Z_s^2 ds < \infty, \int_0^t U_s^2 ds < \infty$, *for all* $t > 0$, *and* $Z \geq b$, $\Gamma^G < 0$, $\Gamma^G \Gamma^P - (\Gamma^{GP})^2 > 0$ *such that the process* $\bar{V}(\Xi)$ *defined by*

$$
\begin{aligned}
d\bar{V}_t(\Xi) = X_t &\left[dP_t + Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma_t^G d\langle G \rangle_t + \tfrac{1}{2}\Gamma_t^P \langle P \rangle_t + \tfrac{1}{2}\Gamma_t^I \langle I \rangle_t \right. \\
&\left. + \Gamma_t^{GI} d\langle G, I \rangle_t + \Gamma_t^{PI} d\langle P, I \rangle_t + \Gamma^{GP} d\langle G, P \rangle_t \right] \\
&+ \bar{\delta}\bar{V}_t(\Xi) dt - H_t dt, \quad \bar{V}_0(\Xi) = \bar{V}_0,
\end{aligned}
$$

where $X_t = -r\bar{\rho}\bar{V}_t(\Xi)$, *satisfies the transversality condition*

$$
\lim_{T \to \infty} \lim_{n \to \infty} \mathbb{E}\left[e^{-\bar{\delta}T \wedge \tau_n} \bar{V}_{T \wedge \tau_n}(\Xi) \right] = 0,
$$

for any sequence of stopping times $\{\tau_n\}_n$ *with* $\lim_n \tau_n = \infty$.

(c) *the class* $\Xi(\Theta)$ *contains a strategy* $\Xi^* = (\bar{c}^*, m^*, Y^*, \bar{y}^*)$ *that maximizes the Hamiltonian, that is, the strategy with*

$$
\begin{aligned}
\bar{c}^* &= (u_A')^{-1}(-r\bar{\rho}\bar{V}(\Xi^*)), \quad m^* = 0, \quad Y^* = -\frac{Z}{\Gamma^G}\Sigma_R^{-1}(\mu - r) - \frac{\Gamma^{GI}}{\Gamma^G}\eta - \frac{\Gamma^{GP}}{\Gamma^G}\bar{y}^*\eta \\
\bar{y}^* &= \frac{\Gamma^{GP}Z - \Gamma^G}{\Gamma^G\Gamma^P - (\Gamma^{GP})^2} \frac{\eta'(\mu - r)}{\eta'\Sigma_R\eta} + \frac{\Gamma^{GI}\Gamma^{PG} - \Gamma^G\Gamma^{PI}}{\Gamma^G\Gamma^P - (\Gamma^{GP})^2};
\end{aligned}
\tag{4.2}
$$

(d) *Denoting the reported portfolio value and the contract value* G^* *and* F^*, *respectively, when Agent employs strategy* Ξ^*, *then, Principal's wealth process, following the dynamics*

$$
dW_t = rW_t dt + dG_t^* + y dI_t - c_t dt - dF_t^*.
\tag{4.3}
$$

satisfies the transversality condition

$$
\lim_{T \to \infty} \lim_{n \to \infty} \mathbb{E}\left[e^{-r\rho W_{T \wedge \tau_n}} \right] = 0,
$$

for any sequence of stopping times $\{\tau_n\}_n$ *with* $\lim_n \tau_n = \infty$.

Similar to Lemmas 3.3 and 3.4, the following results hold.

Lemma 4.2 *Assume that* Σ_R *is invertible and* $\eta'\Sigma_R\eta > 0$. *Consider any Principal's viable strategy* $\Theta = (c, F, y)$. *Then, the strategy* $\Xi^* = (\bar{c}^*, m^*, Y^*, \bar{y}^*)$ *in (4.2) is Agent's optimal strategy in the class* $\Xi(\Theta)$, *and* $\bar{V}_0$ *is Agent's optimal value at time 0.*

Lemma 4.3 *Contract F in any viable strategy satisfies*

$$dF_t = Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma_t^G d\langle G\rangle_t + \tfrac{1}{2}\Gamma_t^P d\langle P\rangle_t + \tfrac{1}{2}\Gamma_t^I d\langle I\rangle_t + \Gamma_t^{GI} d\langle G,I\rangle_t + \Gamma_t^{GP} d\langle G,P\rangle_t + \Gamma_t^{PI} d\langle P,I\rangle_t$$
$$+ \tfrac{1}{2}r\bar{\rho}\, d\langle Z\cdot G + U\cdot I + P\rangle_t - \left(\tfrac{\delta}{r\bar{\rho}} + \bar{H}_t\right)dt,$$

(4.4)

where

$$\bar{H}_t = \tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) - \tfrac{1}{\bar{\rho}} + (Z_t Y_t^* + U_t \eta + \bar{y}^*\eta)'(\mu_t - r)$$
$$+ \tfrac{1}{2}\Gamma_t^G (Y_t^*)'\Sigma_R Y_t^* + \tfrac{1}{2}\Gamma_t^P \bar{y}_t^* \,\eta'\Sigma_R\eta + \tfrac{1}{2}\Gamma_t^I \eta'\Sigma_R\eta$$
$$+ \Gamma_t^{GI} (Y_t^*)'\Sigma_R\eta + \Gamma_t^{GP}\bar{y}_t^* (Y_t^*)'\Sigma_R\eta + \Gamma_t^{PI}\bar{y}_t^* \,\eta'\Sigma_R\eta.$$

(4.5)

In particular, F is adapted to $\mathbf{F}^{G,I,P}$.

Remark 4.4 *The contract in (4.4) depends on the quadratic variation $\langle Z\cdot G + U\cdot I + P\rangle$ of Agent's wealth and quadratic variation $\langle P\rangle$. These are observable by Principal because Agent's wealth is assumed observable by Principal. In the benchmark model of Section 2 in which Agent is not allowed to invest in the index privately, the contract in (3.7) depends only on the quadratic variation $\langle Z\cdot G + U\cdot I\rangle_t$, which is observable to Principal, due to the assumption that $Z, U \in \mathbb{F}^{G,I}$ and Principal observes G and I continuously. Note that in that case $\langle Z\cdot G + U\cdot I\rangle_t$ is also the quadratic variation of Agent's wealth process, but it does not require that Principal can observe Agent's wealth process. Here, instead, we assume Agent's wealth process is observable, which makes the same approach work. It is an interesting open question what the optimal contract is if Agent's wealth process is not observable/contractible and he can invest privately in the index.*

It turns out that with a particular choice of sensitivity processes Γ's, it is optimal for Agent and Principal not to invest in the index. Therefore, the equilibrium is as in the benchmark case.

Theorem 4.5 *Suppose that Assumption 3.7 holds. Then, there exists an equilibrium in which statements in Theorem 3.9, moreover, both Agent and Principal do not invest in the index. The optimal contract in the viable class is*

$$dF_t = C^I dt + \tfrac{\rho}{\rho+\bar{\rho}}dG_t + \xi(dG_t - \beta^{Y^*} dI_t) + \tfrac{r}{2}\zeta d\langle G - \beta^{Y^*} I\rangle_t$$
$$+ \tfrac{1}{2}(\Gamma^P + r\bar{\rho})d\langle P\rangle_t + (\Gamma^{GP} + r\bar{\rho}Z)d\langle G,P\rangle_t + (\Gamma^{PI} + r\bar{\rho}U)d\langle P,I\rangle_t,$$

(4.6)

where Z, U, ξ, ζ are the same as in (3.11), and

$$\Gamma^P < \left(\tfrac{1-Z}{Z}\right)^2\Gamma^G, \quad \Gamma^{GP} = \tfrac{1-Z}{Z}\Gamma^G, \quad \Gamma^{PI} = \tfrac{\Gamma^{GI}}{Z}, \quad \Gamma^G = -rZ\mathscr{C}_b, \quad \Gamma^{GI} = rZ\mathscr{D}_b\beta^{Y^*},$$

$$C^I = -\left(\tfrac{\delta}{r\bar{\rho}} + \bar{H}\right) \quad \text{with } \bar{H} \text{ from (4.5).}$$

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5 Conclusions

We find equilibrium asset prices in a model with OU dynamics for the dividend processes, in a market in which CARA investors hire CARA portfolio managers. The optimal contract involves the quadratic variation of a benchmarked portfolio value which provides incentive to Agent to take on specific risk of individual stocks. We find that the stocks in large supply have high risk premia, and the stocks in low supply have low risk premia, and this effect is stronger as agency friction increases. However, this effect is of a lower order of magnitude than when only the contracts without the quadratic variation terms are allowed, as in BVW (2014). Therefore introducing the quadratic variation term in investor's contracts mitigates the price/return distortion of asset prices in equilibrium. It would be of interest to study, in the future, the problem with dividends modeled as square-root processes, in which case the contract terms would change with the state of the economy, and the volatility would also depend on the agency frictions in equilibrium. This would require numerically solving the corresponding HJB equations. Another open problem is finding the optimal contract when Agent's can hedge and his hedging strategy is not contractible.

6 Proofs

6.1 Proof of Lemma 3.1

We denote $\bar{\mathcal{V}}_t(\Xi)$ by $\bar{\mathcal{V}}_t(\bar{W}_t)$ to emphasize its dependence on $\bar{W}_t$. Let $(\bar{c}'_s)_{s\geq t}$ be Agent's optimal consumption stream from t onwards. Note that $\bar{c}'$ is financed by a wealth process starting from $\bar{W}_t$ at time t. Therefore $\bar{c}' - r\bar{W}_t$ can be financed by a wealth process starting from 0 at time t. Agent's exponential utility function implies that

$$\bar{\mathcal{V}}_t(0) \geq e^{r\bar{\rho}\bar{W}_t}\bar{\mathcal{V}}_t(\bar{W}_t).$$

Above inequality is in fact an equality, i.e., $\bar{c}' - r\bar{W}_t$ is optimal for $\bar{\mathcal{V}}_t(0)$. Assuming otherwise, there exists another consumption stream $\bar{c}''$ whose associated value is strictly larger than $e^{r\bar{\rho}\bar{W}_t}\bar{\mathcal{V}}_t(\bar{W}_t)$. Since $\bar{c}''$ is financed by a wealth process starting from 0 at time t, $\bar{c}'' + r\bar{W}_t$ can be financed by a wealth process starting from $\bar{W}_t$ at time t. Moreover, the expected utility associated to $\bar{c}'' + r\bar{W}_t$ is strictly larger than $\bar{\mathcal{V}}_t(\bar{W}_t)$, contradicting the optimality of $\bar{c}'$ for $\bar{\mathcal{V}}_t(\bar{W}_t)$. Therefore, $\bar{\mathcal{V}}_t(\bar{W}_t) = e^{-r\bar{\rho}\bar{W}_t}\bar{\mathcal{V}}_t(0)$, confirming item (i).

For item (ii), definition of $\bar{\mathcal{V}}_t(\Xi)$ yields

$$\mathbb{E}\big[e^{-\bar{\delta}t}\bar{\mathcal{V}}_t(\Xi)\big] = \mathbb{E}\Big[\int_t^\infty e^{-\bar{\delta}s}u_A(\bar{c}_s^t)ds\Big],$$

where $\bar{c}^t$ is Agent's optimal consumption stream from t onwards. Then, item (ii) follows from applying the monotone convergence theorem on the right-hand side.

6.2 Proof of Lemma 3.3

First order condition for $\bar{c}$ and Y in (3.1) gives

$$u_A'(\bar{c}) = X \quad \text{and} \quad \Gamma^G \Sigma_R Y = -Z(\mu - r) - \Gamma^{GI} \Sigma_R \eta.$$

Since the optimization problem on the right-hand side of (3.1) is concave in $\bar{c}$ and Y, and $Z \geq b$, we have that $\Xi^* = (\bar{c}^*, m^*, Y^*)$ in (3.4) is the optimizer for H.

For an arbitrary Agent's admissible strategy $\Xi = (\bar{c}, m, Y)$, consider the process

$$\tilde{V}_t(\Xi) = \int_0^t e^{-\bar{\delta}s} u_A(\bar{c}_s) ds + e^{-\bar{\delta}t} \bar{V}_t(\Xi), \quad t \geq 0,$$

where $\bar{V}(\Xi)$ is defined via (3.2). The definition of H in (3.1) implies that $\tilde{V}(\Xi)$ is a local supermartingale. Taking a localizing sequence $\{\tau_n\}_n$ for this local supermartingale and arbitrary $T \in \mathbb{R}$, we obtain

$$\mathbb{E}\left[\int_0^{T \wedge \tau_n} e^{-\bar{\delta}s} u_A(\bar{c}_s) ds\right] + \mathbb{E}\left[e^{-\bar{\delta}(T \wedge \tau_n)} \bar{V}_{T \wedge \tau_n}(\Xi)\right] = \mathbb{E}[\tilde{V}_{T \wedge \tau_n}(\Xi)] \leq \tilde{V}_0(\Xi) = \bar{V}_0. \tag{6.1}$$

Sending n, and then T to infinity, applying the monotone convergence theorem to the first term on the left-hand side, and (3.3) to the second term, we obtain

$$\mathbb{E}\left[\int_0^\infty e^{-\bar{\delta}s} u_A(\bar{c}_s) ds\right] \leq \bar{V}_0.$$

For strategy $\Xi^* = (\bar{c}^*, m^*, Y^*)$, $\tilde{V}$ is a local martingale. Then, the inequality in (6.1) is an equality. Sending n, and then T to infinity and using the transversality condition for $\bar{V}(\Xi^*)$, optimality of Ξ^* is confirmed. Thus, $\bar{V}_0$ is Agent's optimal value at time 0. A similar argument works for $\bar{V}_t$.

6.3 Proof of Lemma 3.4

Introduce $\hat{V}_t = \bar{V}_0 e^{-r\bar{\rho}\bar{W}_t}$, where $\bar{W}$ follows (2.5) with F in (3.7). We claim that $\hat{V} = \bar{V}(\Xi)$. Therefore, when Agent is offered the contract F in (3.7), and with everything else remaining the same, his continuation value satisfies the viability condition (3.2). To prove the claim notice first that Hamiltonian H in (3.1) can be written as

$$H = X\left[\frac{1}{\bar{\rho}}\log(X) - \frac{1}{\bar{\rho}} + (ZY^* + U\eta)'(\mu - r) + \frac{1}{2}\Gamma^G(Y^*)'\Sigma_R Y^* + \frac{1}{2}\Gamma^I \eta' \Sigma_R \eta + \Gamma^{GI}(Y^*)'\Sigma_R \eta\right]. \tag{6.2}$$

Next, we also notice that SDE (3.2) for $\bar{V}(\Xi)$ has locally Lipschitz coefficients on $(-\infty, 0)$, hence it admits a unique strong solution before the solution hitting either $-\infty$ or 0. On the other hand,

applying Itô's formula to $\hat{V}$, we have

$$
\begin{aligned}
d\hat{V}_t =& X_t\left[(r\bar{W}_t + bm_t - \bar{c}_t)dt + dF_t\right] - \tfrac{1}{2}r\bar{\rho}X_t d\langle Z\cdot G + U\cdot I\rangle_t \\
=& X_t\left[(r\bar{W}_t + bm_t - \bar{c}_t)dt + Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma_t^G d\langle G\rangle_t + \tfrac{1}{2}\Gamma^I d\langle I\rangle_t + \Gamma_t^{GI} d\langle G,I\rangle_t - \left(\tfrac{\bar{\delta}}{r\bar{\rho}} + \bar{H}_t\right)\right] \\
=& \bar{\delta}\hat{V}_t + X_t\left[dP_t + Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma_t^G d\langle G\rangle_t + \tfrac{1}{2}\Gamma_t^I d\langle I\rangle_t + \Gamma_t^{GI} d\langle G,I\rangle_t - (-r\bar{W}_t + \bar{H}_t)\right] \\
=& \bar{\delta}\hat{V}_t + X_t\left[dP_t + Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma_t^G d\langle G\rangle_t + \tfrac{1}{2}\Gamma_t^I\langle I\rangle_t + \Gamma_t^{GI} d\langle G,I\rangle_t\right] - H_t dt,
\end{aligned}
$$

where $X_t = -r\bar{\rho}\hat{V}_t$ and the fourth identity follows from $-r\bar{W}_t + \tfrac{1}{\rho}\log(-r\bar{\rho}\bar{V}_0) = \tfrac{1}{\rho}\log(-r\bar{\rho}\hat{V}_t) = \tfrac{1}{\rho}\log(X)$ and (6.2). Thus, $\hat{V}$ satisfies (3.2) and it does not hit $-\infty$ or 0 in finite time, since $\bar{W}$ does not hit $-\infty$ nor ∞ in finite time. Therefore, $\hat{V}$ is the unique solution of (3.2).

6.4 Proof of Theorem 3.6

6.4.1 Step 1: Preparation

Given $Z, U, \Gamma^G, \Gamma^{GI}$ satisfying $Z \geq b$ and $\Gamma^G < 0$, we denote

$$
\bar{U} = \tfrac{U}{Z}, \quad \bar{\Gamma}^G = -\tfrac{Z}{\Gamma^G}, \quad \text{and} \quad \bar{\Gamma}^{GI} = -\tfrac{\Gamma^{GI}}{\Gamma^G}. \tag{6.3}
$$

Then, Agent's optimal strategy Ξ^* from (3.4) is

$$
\bar{c}^* = (u_A')^{-1}(-r\bar{\rho}\bar{V}(\Xi^*)), \quad m^* = 0, \quad Y^* = \bar{\Gamma}^G\alpha + \bar{\Gamma}^{GI}\eta, \tag{6.4}
$$

where $\alpha = \{\alpha_t\}_{t \geq 0}$ with $\alpha_t = \Sigma_R^{-1}(\mu_t - r)$.

When Agent employs the optimal strategy Ξ^*, using (3.7) and (3.8), we see that the contract takes the form

$$
dF_t^* = -\left(\tfrac{1}{\rho}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{\bar{\delta}}{r\bar{\rho}} - \tfrac{1}{\rho} - \tfrac{1}{2}r\bar{\rho}Z_t^2(Y_t^* + \bar{U}_t\eta)'\Sigma_R(Y_t^* + \bar{U}_t\eta)\right)dt + Z_t(Y_t^* + \bar{U}_t\eta)'(\gamma dB_t^p + \sigma dB_t^e).
$$

Then, Principal's wealth process (3.5) follows

$$
\begin{aligned}
dW_t =& \left(rW_t - c_t + \tfrac{1}{\rho}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{\bar{\delta}}{r\bar{\rho}} - \tfrac{1}{\rho} - \tfrac{1}{2}r\bar{\rho}Z_t^2(Y_t^* + \bar{U}_t\eta)'\Sigma_R(Y_t^* + \bar{U}_t\eta)\right)dt \\
&+ (Y_t^* + y_t\eta)'dR_t - Z_t(Y_t^* + \bar{U}_t\eta)'(\gamma dB_t^p + \sigma dB_t^e).
\end{aligned} \tag{6.5}
$$

6.4.2 Step 2: Principal's HJB equation

For constant μ, we conjecture that Principal's value function is given as

$$
V(w) = Ke^{-r\rho w}, \tag{6.6}
$$

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for some constant $K < 0$. This value function is expected to satisfy the following HJB equation

$$\delta V = \sup_{Z \geq b, \bar{U}, \bar{\Gamma}^G, \bar{\Gamma}^{GI}, c, y} \left\{ u_P(c) + V_w \left[rw - c + \frac{1}{\bar{\rho}} \log(-r\bar{\rho}\bar{V}_0) + \frac{\delta}{r\bar{\rho}} - \frac{1}{\bar{\rho}} \right] \right.$$
$$+ V_w \left[(Y^* + y\eta)'(\mu - r) - \frac{1}{2} r\bar{\rho} Z^2 (Y^* + \bar{U}\eta)' \Sigma_R (Y^* + \bar{U}\eta) \right]$$
$$\left. + \frac{1}{2} V_{ww} \left[(Y^* + y\eta) - Z(Y^* + \bar{U}\eta) \right]' \Sigma_R \left[(Y^* + y\eta) - Z(Y^* + \bar{U}\eta) \right] \right\}.$$
$$(6.7)$$

Note that $Y^* + y\eta = \bar{\Gamma}^G \alpha + (\bar{\Gamma}^{GI} + y)\eta$ and $Y^* + \bar{U}\eta = \bar{\Gamma}^G \alpha + (\bar{\Gamma}^{GI} + \bar{U})\eta$. Therefore, instead of optimizing over $\bar{U}, \bar{\Gamma}^{GI}$, and y individually, we can optimize over $\bar{\Gamma}^{GI} + y, \bar{U} - y$, and still obtain the same maximum value. This means that we can set

$$y = 0. \tag{6.8}$$

The maximizer of c in (6.7) is

$$c = (u_P')^{-1}(V_w). \tag{6.9}$$

Plugging (6.6), (6.8), and (6.9) back into (6.7), and taking into account that $K < 0$, we reduce (6.7) to

$$r - \delta = \sup_{Z \geq b, \bar{U}, \bar{\Gamma}^{GI}, \bar{\Gamma}^G} \left\{ \frac{\rho}{\bar{\rho}} (\bar{\delta} - r) + r\rho \left[(Y^*)'(\mu - r) + \frac{1}{\bar{\rho}} \log(-r\bar{\rho}\bar{V}_0) + \frac{1}{\rho} \log(-r\rho K) \right] \right.$$
$$- \frac{1}{2} r^2 \rho^2 \left[(Y^*)' \Sigma_R Y^* \right] + r^2 \rho^2 Z \left[(Y^* + \bar{U}\eta)' \Sigma_R Y^* \right]$$
$$\left. - \frac{1}{2} r^2 \rho (\bar{\rho} + \rho) Z^2 \left[(Y^* + \bar{U}\eta)' \Sigma_R (Y^* + \bar{U}\eta) \right] \right\}. \tag{6.10}$$

The first order condition of optimality for $\bar{U}$ in (6.10) yields

$$(\rho + \bar{\rho}) Z [\eta' \Sigma_R \eta] \bar{U} = [\rho - (\rho + \bar{\rho}) Z] [\eta' \Sigma_R Y^*].$$

Since we assume that $\eta' \Sigma_R \eta > 0$, the concavity in $\bar{U}$ of the maximization problem in (6.10) implies that the maximizer in $\bar{U}$ is

$$\bar{U} = \frac{\rho - (\rho + \bar{\rho}) Z}{(\rho + \bar{\rho}) Z} \frac{Covar^{Y^*, \eta}}{Var^\eta}. \tag{6.11}$$

Using (6.4), the first order condition for $\bar{\Gamma}^{GI}$ in (6.10) is

$$0 = \eta'(\mu - r) - r\mathscr{C}_b \left[Covar^{\alpha, \eta} \bar{\Gamma}^G + Var^\eta \bar{\Gamma}^{GI} \right] + rZ(\rho - (\rho + \bar{\rho}) Z) Var^\eta \bar{U}. \tag{6.12}$$

Plugging in (6.11) for $\bar{U}$, the previous equation is transformed into

$$0 = \eta'(\mu - r) + r(\mathscr{D}_b - \mathscr{C}_b) \left[Covar^{\alpha, \eta} \bar{\Gamma}^G + Var^\eta \bar{\Gamma}^{GI} \right], \tag{6.13}$$

where

$$\mathscr{C}_b = \rho(1 - Z)^2 + \bar{\rho} Z^2 \quad \text{and} \quad \mathscr{D}_b = \frac{(\rho - (\rho + \bar{\rho}) Z)^2}{\rho + \bar{\rho}}. \tag{6.14}$$

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Similarly, the first order condition for $\bar{\Gamma}^G$ in (6.10) is

$$0 = \alpha'(\mu - r) - r\mathscr{C}_b\left[Var^\alpha\bar{\Gamma}^G + Covar^{\eta,\alpha}\bar{\Gamma}^{GI}\right] + rZ(\rho - (\rho + \bar{\rho})Z)Covar^{\eta,\alpha}\bar{U}.$$

Plugging in the expression (6.11) for $\bar{U}$, the previous equation is transformed into

$$0 = \alpha'(\mu - r) + r(\mathscr{D}_b - \mathscr{C}_b)Covar^{\eta,\alpha}\bar{\Gamma}^{GI} + r\left[\mathscr{D}_b\frac{(Covar^{\eta,\alpha})^2}{Var^\eta} - \mathscr{C}_bVar^\alpha\right]\bar{\Gamma}^G. \qquad (6.15)$$

Solving (6.13) and (6.15) for $\bar{\Gamma}^G$ and $\bar{\Gamma}^{GI}$, and using

$$Covar^{\eta,\alpha} = \eta'\Sigma_R\alpha = \eta'\Sigma_R\Sigma_R^{-1}(\mu - r) = \eta'(\mu - r),$$

$$Var^\alpha = \alpha'\Sigma_R\alpha = \alpha'(\mu - r),$$

we obtain

$$\bar{\Gamma}^G = \frac{1}{r\mathscr{C}_b}, \qquad (6.16)$$

$$\bar{\Gamma}^{GI} = \frac{\mathscr{D}_b}{r\mathscr{C}_b(\mathscr{C}_b - \mathscr{D}_b)}\frac{\eta'(\mu - r)}{Var^\eta}. \qquad (6.17)$$

On the right-hand side of (6.10), the function to be maximized tends to negative infinity when either $|\bar{\Gamma}^G| \to \infty$ or $|\bar{\Gamma}^{GI}| \to \infty$. Therefore, $\bar{\Gamma}^G$ and $\bar{\Gamma}^{GI}$ obtained in (6.16) and (6.17) are the maximizers for the maximization problem in (6.10). Moreover, since $\mu - r$ is a constant vector, $Y^* = \bar{\Gamma}^G\alpha + \bar{\Gamma}^{GI}\eta$ is a constant vector as well.

Another form of $\bar{\Gamma}^{GI}$ that will be useful later can be obtained by plugging (6.16) back into (6.12) and using (6.11). This gives

$$\bar{\Gamma}^{GI} = \frac{\mathscr{D}_b}{\mathscr{C}_b}\frac{Covar^{Y^*,\eta}}{Var^\eta}. \qquad (6.18)$$

Finally, the unconstrained first order condition for Z in (6.10) gives

$$0 = \rho\left[(Y^* + \bar{U}\eta)'\Sigma_RY^*\right] - (\rho + \bar{\rho})Z\left[(Y^* + \bar{U}\eta)'\Sigma_R(Y^* + \bar{U}\eta)\right].$$

Plugging the expression of $\bar{U}$ from (6.11) into the previous equation, we can solve it and get $Z = \frac{\rho}{\rho + \bar{\rho}}$. Since the maximization problem in (6.7) is concave in Z, under the constraint $Z \geq b$ optimal Z is

$$Z = \max\left\{\frac{\rho}{\rho + \bar{\rho}}, b\right\}. \qquad (6.19)$$

6.4.3 Step 3: Optimal contract

Plugging (6.11), (6.16), and (6.18) back to (6.3) yields

$$U = -(Z - \frac{\rho}{\rho + \bar{\rho}})\beta^{Y^*}, \quad \Gamma^G = -rZ\mathscr{C}_b, \quad \text{and} \quad \Gamma^{GI} = rZ\mathscr{D}_b\beta^{Y^*}, \quad \text{where} \quad \beta^{Y^*} = \frac{Covar^{Y^*,\eta}}{Var^\eta}.$$

Combining the previous expressions with (3.7), we obtain

$$ZdG_t + U_t dI_t = \frac{\rho}{\rho+\bar{\rho}}dG_t + \xi(dG_t - \beta^{Y^*}dI_t),$$

$$\tfrac{1}{2}\Gamma^G d\langle G\rangle_t + \Gamma^{GI}d\langle G,I\rangle_t + \tfrac{1}{2}r\bar{\rho}Z^2 d\langle G+\bar{U}I\rangle_t = \tfrac{r}{2}\zeta\left[d\langle G\rangle_t - 2d\langle G,\beta^{Y^*}I\rangle_t\right] + \tfrac{1}{2}r\bar{\rho}U^2 d\langle I\rangle_t,$$

where

$$\xi = (b - \tfrac{\rho}{\rho+\bar{\rho}})_+ \quad \text{and} \quad \zeta = (\rho+\bar{\rho})Z(1-Z)(b - \tfrac{\rho}{\rho+\bar{\rho}})_+.$$

In order to have $\langle G - \beta^{Y^*}I\rangle_t$ instead of $\langle G\rangle_t - 2\langle G,\beta^{Y^*}I\rangle_t$ in the above expression, we introduce

$$\tfrac{1}{2}\Gamma^I = \tfrac{r}{2}\left[\xi - \bar{\rho}(Z - \tfrac{\rho}{\rho+\bar{\rho}})^2\right](\beta^{Y^*})^2.$$

Then,

$$\tfrac{1}{2}\Gamma^G d\langle G\rangle_t + \Gamma^{GI}d\langle G,I\rangle_t + \tfrac{1}{2}\Gamma^I d\langle I\rangle_t + \tfrac{1}{2}r\bar{\rho}Z^2 d\langle G+\bar{U}I\rangle_t = \tfrac{r}{2}\zeta d\langle G - \beta^{Y^*}I\rangle_t.$$

On the other hand,

$$\frac{\bar{\delta}}{r\bar{\rho}} + \bar{H} = \tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{\bar{\delta}}{r\bar{\rho}} - \tfrac{1}{\bar{\rho}} + (ZY^* + U\eta)'(\mu - r)$$
$$+ \tfrac{r}{2}\zeta(Y^* - \beta^{Y^*}I)'\Sigma_R(Y^* - \beta^{Y^*}I) - \tfrac{r}{2}\bar{\rho}(ZY^* + U\eta)'\Sigma_R(ZY^* + U\eta).$$

Collecting above results and combining them with (3.7), we obtain

$$dF_t = Cdt + \frac{\rho}{\rho+\bar{\rho}}dG_t + \xi(dG_t - \beta^{Y^*}dI_t) + \tfrac{r}{2}\zeta d\langle G - \beta^{Y^*}I\rangle_t, \tag{6.20}$$

where

$$C = -\tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) - \tfrac{\bar{\delta}}{r\bar{\rho}} + \tfrac{1}{\bar{\rho}} - (ZY^* + U\eta)'(\mu - r)$$
$$- \tfrac{r}{2}\zeta(Y^* - \beta^{Y^*}I)'\Sigma_R(Y^* - \beta^{Y^*}I) + \tfrac{r}{2}\bar{\rho}(ZY^* + U\eta)'\Sigma_R(ZY^* + U\eta). \tag{6.21}$$

6.4.4 Step 4: Verifications

Let us verify that $\Theta = (c, F, y)$, defined by (6.9), (6.20), and (6.8), is viable. First, when Agent employs the strategy Ξ^* with constant Y^*, we have from (3.2), (3.1) that

$$d\bar{V}_t(\Xi^*) = -r\bar{\rho}\bar{V}_t(\Xi^*)[ZY^* + U\eta]'(\gamma dB_t^p + \sigma dB_t^e) + (\bar{\delta} - r)\bar{V}_t(\Xi^*)dt.$$

Therefore, $\bar{V}(\Xi^*)$ is given by

$$\bar{V}_t(\Xi^*) = \bar{V}_0 e^{(\bar{\delta}-r)t}\mathscr{E}\left(-r\bar{\rho}\int_0^t (ZY^* + U\eta)'(\gamma dB_s^p + \sigma dB_s^\sigma)\right).$$

We need to show that $\bar{V}(\Xi^*)$ satisfies the transversality condition (3.3). To this end, take any $T \in \mathbb{R}$ and any sequence of stopping times $\{\tau_n\}_n$ converging to infinity. Since Y^* is a constant vector,

then Z and U are constants, therefore the above stochastic exponential is a martingale, hence the family

$$\left\{ \mathscr{E}\left(-r\bar{\rho} \int_0^{T\wedge\tau_n} (Z\theta + U\eta)'(\gamma dB_s^p + \sigma dB_s^\sigma) \right) \right\}_n \quad \text{is uniformly integrable in } n.$$

As a result,

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-\bar{\delta}T\wedge\tau_n} \bar{V}_{T\wedge\tau_n}(\Xi^*) \right] = \bar{V}_0 e^{-rT} \mathbb{E}\left[\mathscr{E}\left(-r\bar{\rho} \int_0^T (Z\theta + U\eta)'(\gamma dB_s^p + \sigma dB_s^\sigma) \right) \right] = \bar{V}_0 e^{-rT},$$

which vanishes when $T \to \infty$. Therefore Lemma 3.3 shows that $\Xi^* = (\bar{c}^*, m^*, Y^*)$ in (3.4) is Agent's optimal strategy in $\Xi(\Theta)$.

Next we need to show that Θ is adapted to $\mathbf{F}^{G^*,I}$. Combining (6.6) and (6.9) yields

$$c = -\tfrac{1}{\rho}\log(-r\rho K) + rW.$$

Plugging this expression for c into (3.5) we obtain

$$dW_t = \tfrac{1}{\rho}\log(-r\rho K)dt + dG_t^* - dF^*.$$

We have seen in Lemma 3.4 that F^* is adapted to $\mathbf{F}^{G^*,I}$, thus W and c are adapted to the same filtration.

It remains to check the transversality condition (3.6) is satisfied. To this end, applying Itô's formula to $V_t = V(W_t)$, and using (6.7) and (6.9), we obtain

$$dV_t = (\delta - r)V_t dt - r\rho V_t [Y^* - (ZY^* + U\eta)]'[\gamma dB_t^p + \sigma dB_t^e].$$

Therefore

$$V_t = V_0 e^{(\delta-r)t} \mathscr{E}\left(-r\rho \int_0^t [Y^* - (ZY^* + U\eta))]'[\gamma dB_s^p + \sigma dB_s^e] \right).$$

The same argument leading to verify (3.3) above yields that (3.6) is satisfied. This concludes the proof of viability for Principal's strategy Θ.

Let us now verify the optimality of Principal's strategy Θ. For arbitrary Principal's viable strategy $\tilde{\Theta} = (\tilde{c}, \tilde{F}, \tilde{y})$ and its associated Agent's optimal strategy $\tilde{\Xi}^*$, consider the process

$$\tilde{V}_t = \int_0^t e^{-\delta s} u_P(\tilde{c}_s)ds + e^{-\delta t}V(W_t),$$

where $V(w)$ is defined in (6.6). From the HJB equation (6.7), we obtain that $\tilde{V}$ is a local supermartingale. Using the same localization argument as in the proof of Lemma 3.3 together with the transversality condition (3.6), we obtain

$$\mathbb{E}\left[\int_0^\infty e^{-\delta s} u_P(\tilde{c}_s)ds \right] \le \tilde{V}_0 = V(W_0),$$

where the inequality is equality when Principal chooses Θ. This verifies the optimality of Θ.

6.5 Proof of Theorem 3.9

6.5.1 Step 1: Equilibrium asset prices

In equilibrium, given that $y = 0$, we necessarily have $Y^* = \theta$. Then, (6.4) combined with (6.16) and (6.18) yields

$$\theta = \frac{1}{r\mathscr{C}_b}\alpha_t + \frac{\mathscr{D}_b}{\mathscr{C}_b}\beta^\theta\eta. \tag{6.22}$$

Recall that $\alpha_t = \Sigma_R^{-1}(\mu_t - r)$. Left-multiplying (6.22) by $r\mathscr{C}_b\eta'\Sigma_R$ leads to

$$r\mathscr{C}_b\eta'\Sigma_R\theta = \eta'(\mu_t - r) + r\mathscr{D}_bCovar^{\theta,\eta}.$$

Note that all the terms above equation are constants except for the first term on the right-hand side, which is $\eta'(p_tA_1 + e_tA_2 + A_3)$. Since this equation has to hold for all values of p_t and e_t in equilibrium, it is necessary to have

$$A_1 = 0 \quad \text{and} \quad A_2 = 0.$$

Hence μ is a constant. Recalling the definition of A_1 and A_2 in (2.12), we then obtain

$$a_{pi} = \frac{a_i}{r + \kappa^p}, \quad a_{ei} = \frac{1}{r + \kappa_i^e}, \quad i = 1, \dots, N. \tag{6.23}$$

In order to determine a_0, we left-multiply both sides of (6.22) by $r\mathscr{C}_b\Sigma_R$, and using $A_1 = A_2 = 0$ it follows that

$$A_3 = \mu - r = r\Sigma_R\left(\mathscr{C}_b\theta - \mathscr{D}_b\beta^\theta\eta\right). \tag{6.24}$$

Note that $\mathscr{C}_b - \mathscr{D}_b = \frac{\rho\bar{\rho}}{\rho+\bar{\rho}}$. Thus, the previous equation can be rewritten as

$$A_3 = \mu - r = r\Sigma_R\left(\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}\theta + \mathscr{D}_b(\theta - \beta^\theta\eta)\right). \tag{6.25}$$

Recalling the definition of A_3 from (2.12), (6.25) yields

$$a_0 = \frac{1}{r}\kappa^p\bar{p}a_p + \frac{1}{r}(\kappa^e)'\bar{e}a_e - \frac{\rho\bar{\rho}}{\rho+\bar{\rho}}\Sigma_R\theta - \mathscr{D}_b\Sigma_R(\theta - \beta^\theta\eta). \tag{6.26}$$

Plugging (6.16), (6.17), and $\mathscr{C}_b - \mathscr{D}_b = \frac{\rho\bar{\rho}}{\rho+\bar{\rho}}$ back to (6.4), we confirm (3.12). Left-multiplying both sides of (6.25) by θ', we obtain the excess return of the portfolio:

$$\theta'(\mu - r) = r\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}Var^\theta + r\mathscr{D}_b\left(Var^\theta - \frac{(Covar^{\theta,\eta})^2}{Var^\eta}\right). \tag{6.27}$$

Left-multiplying η' both sides of (6.25) by η', we obtain the excess return of the index:

$$\eta'(\mu - r) = r\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}Covar^{\theta,\eta}. \tag{6.28}$$

Finally, since a_{pi} and a_{ei} obtained in (6.23) are positive, all entries of Σ_R are positive. Moreover all entries of η are positive. Therefore $\eta'\Sigma_R\eta > 0$ is also confirmed.

6.5.2 Step 2: Participation constraint and Principal's value

We now determine Principal's optimal choice of Agent's value at time 0, i.e., $\bar{V}_0$, so that Agent is willing to take this contract. and that Principal is willing to issue the contract F.

If Agent does not take the contract, his value function $\bar{V}^u$ is expected to satisfy the following HJB equation

$$\bar{\delta}\bar{V}^u = \sup_{\bar{c}^u,Y}\left\{u_A(\bar{c}^u) + \bar{V}^u_{\bar{w}}(r\bar{w} + Y'(\mu - r) - \bar{c}^u) + \tfrac{1}{2}\bar{V}^u_{\bar{w}\bar{w}}Y'\Sigma_R Y\right\}, \tag{6.29}$$

We conjecture that $\bar{V}^u$ takes the form

$$\bar{V}^u(\bar{w}) = \bar{K}^u e^{-r\bar{\rho}\bar{w}},$$

for some constant $\bar{K}^u < 0$. The first order conditions for the maximization of $\bar{c}^u$ and Y give

$$\bar{c}^u = -\tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{K}^u) + r\bar{w},$$

$$Y = \tfrac{1}{r\bar{\rho}}\Sigma_R^{-1}(\mu - r),$$

which are optimizers for the right-hand side of (6.29) due to concavity. Plugging above $\bar{c}^u$ and Y back to (6.29) yields

$$\log(-r\bar{\rho}\bar{K}^u) = \tfrac{r-\bar{\delta}}{r} - \tfrac{1}{2r}(\mu - r)'\Sigma_R^{-1}(\mu - r). \tag{6.30}$$

An argument similar to the proof of Lemma 3.3 verifies the optimality of $(\bar{c}^u, Y)$. Since $\bar{W}_0 = 0$, Principal can set $\bar{V}_0 = \bar{K}^u$. In this case, Agent is indifferent with respect to taking the contract or not, in which case we assume he chooses to work for Principal. Plugging (6.30) into (6.21) yields

$$\begin{aligned}
C =&\, \tfrac{1}{2r}\tfrac{1}{\bar{\rho}}(\mu - r)'\Sigma_R^{-1}(\mu - r) - (ZY^* + U\eta)'(\mu - r) \\
&- \tfrac{r}{2}\zeta(Y^* - \beta^{Y^*}I)'\Sigma_R(Y^* - \beta^{Y^*}I) + \tfrac{r}{2}\bar{\rho}(ZY^* + U\eta)'\Sigma_R(ZY^* + U\eta).
\end{aligned} \tag{6.31}$$

Let us determine K in (6.6). First, plugging $\mu - r$ from (6.24) into the right-hand side of (6.30), we obtain

$$\log(-r\bar{\rho}\bar{V}_0) = \tfrac{r-\bar{\delta}}{r} - \tfrac{r}{2}\left[\mathscr{C}_b^2 Var^\theta + (\mathscr{D}_b^2 - 2\mathscr{C}_b\mathscr{D}_b)\tfrac{(Covar^{\theta,\eta})^2}{Var^\eta}\right]. \tag{6.32}$$

Second, plugging (6.11) back into (6.10) and using $Y^* = \theta$, we obtain

$$\tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{1}{\rho}\log(-r\rho K) + \theta'(\mu - r) = \tfrac{1}{\rho} + \tfrac{1}{\bar{\rho}} - \tfrac{\delta}{r\rho} - \tfrac{\bar{\delta}}{r\bar{\rho}} + \tfrac{r}{2}\mathscr{C}_b Var^\theta - \tfrac{r}{2}\mathscr{D}_b\tfrac{(Covar^{\theta,\eta})^2}{Var^\eta}. \tag{6.33}$$

Plugging (6.27) and (6.32) back to (6.33), we obtain

$$\tfrac{1}{\rho}\log(-r\rho K) = \tfrac{1}{\rho} - \tfrac{\delta}{r\rho} + \tfrac{r}{2}\tfrac{1}{\bar{\rho}}(\mathscr{C}_b^2 - \bar{\rho}\mathscr{C}_b)Var^\theta + \tfrac{r}{2}\tfrac{1}{\bar{\rho}}(\mathscr{D}_b^2 - 2\mathscr{C}_b\mathscr{D}_b + \bar{\rho}\mathscr{D}_b)\tfrac{(Covar^{\theta,\eta})^2}{Var^\eta}. \tag{6.34}$$

If Principal does not hire Agent, her value function V^u is expected to satisfy the following HJB equation

$$\delta V^u = \sup_{c^u,y}\left\{u_P(c^u) + V^u_w(rw + y\eta'(\mu - r) - c^u) + \tfrac{1}{2}V^u_{ww}y^2\eta'\Sigma_R\eta\right\}. \tag{6.35}$$

We conjecture that V^u takes the form

$$V^u(w) = K^u e^{-r\rho w},$$

for some constant $K^u < 0$. The first order conditions for the maximization of c^u and y give

$$c^u = -\frac{1}{\rho}\log(-r\rho K^u) + rw,$$

$$y = \frac{1}{r\rho}\frac{\eta'(\mu - r)}{\eta'\Sigma_R\eta},$$

which are optimizers for the right-hand side of (6.35) due to concavity. Plugging above c^u and y back to (6.35), we obtain

$$\frac{1}{\rho}\log(-r\rho K^u) = \frac{1}{\rho} - \frac{\delta}{r\rho} - \frac{1}{2r}\frac{1}{\rho}\frac{(\eta'(\mu - r))^2}{\eta'\Sigma_R\eta}. \tag{6.36}$$

Using $\eta'(\mu - r)$ from (6.28), we obtain

$$\frac{1}{\rho}\log(-r\rho K^u) = \frac{1}{\rho} - \frac{\delta}{r\rho} - \frac{r}{2}\frac{1}{\rho}\left(\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}\right)^2\frac{(Covar^{\theta,\eta})^2}{Var^\eta}. \tag{6.37}$$

An argument similar to the proof of Lemma 3.3 verifies the optimality of (c^u, y).

Comparing (6.34) and (6.37), we see that $V(W_0) \geq V^u(W_0)$ if and only if

$$C_3 Var^\theta + C_4\frac{(Covar^{\theta,\eta})^2}{Var^\eta} \geq \frac{r}{2}\left(\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}\right)^2\frac{(Covar^{\theta,\eta})^2}{Var^\eta}, \tag{6.38}$$

where constants C_3 and C_4 are

$$C_3 = \frac{r}{2}\frac{\rho}{\bar{\rho}}\mathscr{C}_b(\bar{\rho} - \mathscr{C}_b) \quad \text{and} \quad C_4 = \frac{r}{2}\frac{\rho}{\bar{\rho}}\mathscr{D}_b(2\mathscr{C}_b - \mathscr{D}_b - \bar{\rho}).$$

Note that $C_3 = \frac{r}{2}\left(\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}\right)^2 - C_4$ and $C_3 \geq 0$ when $0 \leq b \leq 1$. Then, (6.38) is equivalent to $Var^\theta \geq \frac{(Covar^{\theta,\eta})^2}{Var^\eta}$, which holds by Cauchy-Schwarz inequality. Therefore, $V(W_0) \geq V^u(W_0)$ and Principal is willing to hire Agent.

6.6 Calculation for Figure 4

Given the excess return in BVW (2014)

$$\mu - r = r\bar{\rho}\Sigma_R(Z\theta + U\eta), \tag{6.39}$$

where Z and U are as in Theorem 3.6 item (a), and Agent's optimal holding $Y^* = \theta$, Principal's value function is

$$V_{BVW}(W_0) = -e^{-r\rho W_0 - C_{BVW}},$$

where

$$C_{BVW} = -1 + \frac{\delta}{r} + \log(r\rho) - \frac{r}{2}\left(\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}\right)^2\theta'\Sigma_R\theta \\ - r\rho\left(b - \frac{\rho}{\rho+\bar{\rho}}\right)_+ + \left[\frac{\bar{\rho}^2}{\rho+\bar{\rho}} - (\bar{\rho} + \frac{1}{2}\rho)(b - \frac{\rho}{\rho+\bar{\rho}})_+\right]\left(Var^\theta - \frac{(Covar^{\eta,\theta})^2}{Var^\eta}\right). \tag{6.40}$$

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The term $\log(r\rho)$ in above eqaution corresponds to the term $\log(r)$ in BVW (2014), Equation (A.104). The difference is due to Principal's utility being equal to $-e^{-\rho c}$ in BVW (2014), while being equal to $-\frac{1}{\rho}e^{-\rho c}$ in the present paper.

Taking the excess return (6.39), we want to calculate Principal's value if she uses our contract. First, (6.39) yields

$$\eta'(\mu - r) = r\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}Covar^{\eta,\theta}, \quad \alpha = r\bar{\rho}(Z\theta + U\eta).$$

Plugging above two identities back into (6.4), and using $\bar{\Gamma}^G$ and $\bar{\Gamma}^{GI}$ from (6.16) and (6.17), we obtain

$$Y^* = \frac{\bar{\rho}Z}{\mathscr{C}_b}\theta + \frac{\mathscr{D}_b - \bar{\rho}(b - \frac{\rho}{\rho+\bar{\rho}})_+}{\mathscr{C}_b}\beta^\theta\eta. \tag{6.41}$$

Using the above expression and the fact that $\mathscr{C}_b = \frac{\rho\bar{\rho}}{\rho+\bar{\rho}} + \mathscr{D}_b$, we have

$$Covar^{Y^*,\eta} = Covar^{\theta,\eta}.$$

Hence (6.11) yields

$$\bar{U} = \frac{\rho - (\rho+\bar{\rho})Z}{(\rho+\bar{\rho})Z}\beta^\theta.$$

Plugging the previous expression of $\bar{U}$ back into (6.10), a calculation shows that

$$\frac{1}{\rho}\log(-r\bar{\rho}\bar{V}_0) + \frac{1}{\rho}\log(-r\rho K) + (Y^*)'(\mu - r) = \frac{1}{\rho} + \frac{1}{\bar{\rho}} - \frac{\delta}{r\rho} - \frac{\bar{\delta}}{r\bar{\rho}} + \frac{r}{2}\mathscr{C}_bVar^{Y^*} - \frac{r}{2}\mathscr{D}_b\frac{(Covar^{\eta,Y^*})^2}{Var^\eta}.$$

Combining (6.39), (6.41), and the fact that $\mathscr{C}_b = \frac{\rho\bar{\rho}}{\rho+\bar{\rho}} + \mathscr{D}_b$, we show that

$$\frac{r}{2}\mathscr{C}_bVar^{Y^*} - \frac{r}{2}\mathscr{D}_b\frac{(Covar^{\eta,Y^*})^2}{Var^\eta} = \frac{1}{2}(Y^*)'(\mu - r)$$

$$= \frac{r}{2}\frac{\bar{\rho}^2Z^2}{\mathscr{C}_b}Var^\theta + \frac{r}{2}\frac{-\bar{\rho}(b - \frac{\rho}{\rho+\bar{\rho}})_+\mathscr{C}_b + \bar{\rho}Z\mathscr{D}_b - \bar{\rho}^2(b - \frac{\rho}{\rho+\bar{\rho}})_+Z}{\mathscr{C}_b}\frac{(Covar^{\theta,\eta})^2}{Var^\eta}.$$

On the other hand, using (6.30), (6.39), and $\bar{K}^u = \bar{V}_0$, we obtain

$$\log(-r\bar{\rho}\bar{V}_0) = 1 - \frac{\bar{\delta}}{r} - \frac{r}{2}\bar{\rho}^2Z^2Var^\theta - \frac{r}{2}\left[-2Z(b - \frac{\rho}{\rho+\bar{\rho}}) + \bar{\rho}^2 + (b - \frac{\rho}{\rho+\bar{\rho}})_+^2\bar{\rho}^2\right]\frac{(Covar^{\theta,\eta})^2}{Var^\eta}.$$

Combining the previous three equations, Principal's value, when she employs the contract in (3.10), is

$$V_{CX}(W_0) = Ke^{-r\rho W_0} = -e^{-r\rho W_0 - C_{CX}},$$

where

$$C_{CX} = -1 + \frac{\delta}{r} + \log(r\rho)$$

$$+ \frac{r}{2}\frac{\rho\bar{\rho}Z^2}{\mathscr{C}_b}\left[\frac{\bar{\rho}^2}{\rho+\bar{\rho}} - (\rho+\bar{\rho})(b - \frac{\rho}{\rho+\bar{\rho}})_+^2\right]Var^\theta$$

$$+ \frac{r}{2}\left[\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}(\rho - \bar{\rho})(b - \frac{\rho}{\rho+\bar{\rho}})_+ + \rho\bar{\rho}(b - \frac{\rho}{\rho+\bar{\rho}})_+^2 + \frac{\rho\bar{\rho}Z}{\mathscr{C}_b}(\mathscr{D}_b - \bar{\rho}(b - \frac{\rho}{\rho+\bar{\rho}})_+)\right]\frac{(Covar^{\theta,\eta})^2}{Var^\eta}. \tag{6.42}$$

Figure 4 compares the certainty equivalences C_{BVW}/ρ and C_{CX}/ρ for Principal under the two contracts.

6.7 Proof of Theorem 4.5

When Agent employs the optimal strategy Ξ^* in (4.2), using (4.4), (4.5), and noticing that (4.4) does not contain $\bar{y} \cdot I$, we obtain

$$dF^* = -\left[\tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{\bar{\delta}}{r\bar{\rho}} - \tfrac{1}{\bar{\rho}} + \bar{y}^*\eta'(\mu - r) - \tfrac{1}{2}r\bar{\rho}(Z_tY_t^* + U_t\eta + \bar{y}^*\eta)'\Sigma_R(Z_tY^* + U_t\eta + \bar{y}^*\eta)\right]dt$$
$$+ (Z_tY_t^* + U_t\eta)'(\gamma dB_t^p + \sigma dB_t^e).$$

Therefore, Principal's wealth process in (4.3) follows

$$dW_t = \left[rW_t - c_t + \tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{\bar{\delta}}{r\bar{\rho}} - \tfrac{1}{\bar{\rho}}\right]dt$$
$$+ \left[(Y_t^* + y_t\eta + \bar{y}_t^*\eta)'(\mu - r) - \tfrac{1}{2}r\bar{\rho}(Z_tY_t^* + U_t\eta + \bar{y}_t^*\eta)'\Sigma_R(Z_tY^* + U_t\eta + \bar{y}_t^*\eta)\right]dt$$
$$+ \left[(Y_t^* + y_t\eta) - (Z_tY_t^* + U_t\eta)\right]'(\gamma dB_t^p + \sigma dB_t^e).$$

We conjecture that Principal's value function is given by

$$V(w) = Ke^{-r\rho w}, \tag{6.43}$$

for some constant $K < 0$. The value function is expected to satisfy the following HJB equation

$$\delta V = \sup_{Z \geq b, U, \Gamma's, c, y}\left\{u_p(c) + V_w\left[rw - c + \tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) + \tfrac{\bar{\delta}}{r\bar{\rho}} - \tfrac{1}{\bar{\rho}}\right]\right.$$
$$+ V_w\left[(Y^* + y\eta + \bar{y}^*\eta)'(\mu - r) - \tfrac{1}{2}r\bar{\rho}(ZY^* + U\eta + \bar{y}^*\eta)'\Sigma_R(ZY^* + U\eta + \bar{y}^*\eta)\right]$$
$$\left.+ \tfrac{1}{2}V_{ww}\left[(Y^* + y\eta) - (ZY^* + U\eta)\right]'\Sigma_R\left[(Y^* + y\eta) - (ZY^* + U\eta)\right]\right\}. \tag{6.44}$$

Recalling (4.2), we have

$$Y^* + y\eta + \bar{y}^*\eta = -\tfrac{Z}{\Gamma^G}\alpha - \left(\tfrac{\Gamma^{GI}}{\Gamma^G} - y\right)\eta - \left(\tfrac{\Gamma^{GP}}{\Gamma^G} - 1\right)\bar{y}^*\eta,$$
$$ZY^* + U\eta + \bar{y}^*\eta = Z(Y^* + y\eta + \bar{y}^*\eta) + (U - Zy - (Z-1)\bar{y}^*)\eta,$$
$$(Y^* + y\eta) - (ZY^* + U\eta) = (Y^* + y\eta + \bar{y}^*\eta) - (ZY^* + U\eta + \bar{y}^*\eta).$$

Therefore, instead of optimizing over $Z, \bar{U}, \Gamma'$ s, $\bar{y}$ and y individually, we can optimize over Z, $\Gamma^{GI} - y\Gamma^G$, $\Gamma^{GP} - \Gamma^G$ and $U - Zy - (Z-1)\bar{y}^*$. This means that we can set $y + \bar{y}^* = 0$ and $\bar{y}^* = 0$, i.e.,

$$y = \bar{y}^* = 0. \tag{6.45}$$

Using this and (6.45), we have from (4.2) that

$$Y^* = -\frac{Z}{\Gamma^G}\alpha - \frac{\Gamma^{GI}}{\Gamma^G}\eta,$$
$$\bar{y}^* = \frac{\Gamma^{GP}Z - (1-Z)\Gamma^G}{\Gamma^G\Gamma^P - (\Gamma^{GP})^2}\frac{\eta'(\mu - r)}{\eta'\Sigma_R\eta} + \frac{\Gamma^{GI}(\Gamma^{PG} + \Gamma^G) - \Gamma^G\Gamma^{PI}}{\Gamma^G\Gamma^P - (\Gamma^{GP})^2}.$$

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For given Z, Γ^G, Γ^{GI} with $Z > 0, \Gamma^G < 0$, we can take

$$\Gamma^{GP} = \tfrac{1-Z}{Z}\Gamma^G, \quad \Gamma^P < \left(\tfrac{1-Z}{Z}\right)^2\Gamma^G, \quad \Gamma^{PI} = \tfrac{\Gamma^{GI}}{Z}.$$

Then $\Gamma^G\Gamma^P - (\Gamma^{GP})^2 > 0$ and $\bar{y}^* = 0$ are satisfied. This reduces to the case in which Agent is not allowed to invest in the index privately. Hence, the remainder of the proof is the same as before.

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