$\boxed{\text{Lecture 1}}$

I. Gravity as QFT of spin-2 massless particles

A. Charge conservation and equivalence principle from scattering amplitudes

1) Massless representations of the Poincare group

$$\mathcal{P} = SO(1,3) \ltimes \mathcal{T}$$

traslations in 4d

$$[P_\mu, P_\nu] = 0$$

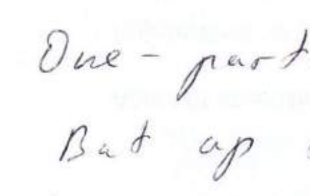
Lorentz gr.
preserves
$-x_0^2 + x_i^2$

N.B. signature!

One-particle states $|\psi\rangle$ form representations of $\mathcal{P}$

But up to phase

$\Rightarrow$ true representations of the universal covering



$$\overline{\mathcal{P}} = \underline{SL(2)} \ltimes \mathcal{T}.$$

2x2 complex matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$

covers $SO(3)$ with $SU(2)$

<u>Exercise 1</u> prove that $SL(2)$ is universal covering of $SO(1,3)$. Identify the covering map

1a) Representations of Lorentz group
Generators:

$$J_i = \frac{\sigma_i}{2} \quad ; \quad K_i = i\frac{\sigma_i}{2}$$

space-rotations

$$A \in sl(2) \Rightarrow A = \sum \alpha_i J_i + \beta_i K_i$$

real

boosts

$$[J_i, J_j] = i\varepsilon_{ijk} J_k$$

$$[J_i, K_j] = i\varepsilon_{ijk} K_k$$

$$[K_i, K_j] = -i\varepsilon_{ijk} J_k$$

$$N_i^\pm = \frac{J_i \pm iK_i}{2}$$

$$\Rightarrow \left.\begin{array}{l} [N_i^\pm, N_j^\pm] = i\varepsilon_{ijk} N_k^\pm \\ [N_i^+, N_j^-] = 0 \end{array}\right\} \quad 2 \text{ copies of } su(2)$$

Finite-dimensional reps.

$$(j_+, j_-)$$

half-integers

$$J_i = N_i^+ + N_i^- \Rightarrow j = j_+ + j_-, j_+ + j_- - 1, \ldots |j_+ - j_-|$$

$(0,0)$ - scalar

$(\frac{1}{2},0)$, $(0,\frac{1}{2})$ - (Weyl) spinors

$(\frac{1}{2},\frac{1}{2})$ - vector
$$\begin{tikzcd} & A_i \\ A_\mu \arrow[ur] \arrow[dr] & \\ & A_0 \end{tikzcd}$$

$(1,0)$ - self-dual form $F^{\pm}_{\mu\nu} = \pm\frac{i}{2}\varepsilon_{\mu\nu\lambda\rho}F^{\pm\lambda\rho}$

$(0,1)$ - anti-self-dual

$(1,1)$ - tensor $h_{\mu\nu}$

$(2,0)$ - $R^{+}_{\mu\nu\lambda\rho} = \frac{i}{2}\varepsilon_{\mu\nu\mu'\nu'}R^{+\mu'\nu'}{}_{\lambda\rho}$

$$R^{+}_{\mu\nu\lambda\rho} = R^{+}_{\lambda\rho\mu\nu}$$

$$\eta_{\mu\lambda}R^{+}_{\mu\nu\lambda\rho} = 0$$

<u>Ex. 2</u> Prove that $R^{+}_{\mu\nu\lambda\rho}$ has 5 independent components

10) Little group

$SO(1,3) \ltimes T$

$P_\mu |\psi\rangle = k_\mu |\psi\rangle$

$[P_\mu^2, A] = 0 \Rightarrow k_\mu^2 = -m^2$ fixed

$m \neq 0 \Rightarrow$ go to the frame $k^0 = m$, $k^i = 0$

$L^\mu{}_\nu$ – leave $\bar{k}_\mu$ invariant – rotations in 3d space

$U(L^\mu{}_\nu)|\psi\rangle \leftarrow$ unitary reps of $SU(2)$

$|\sigma, k\rangle \equiv \underbrace{U(\Lambda^\mu{}_\nu)}_{} |\sigma\rangle$

~~takes~~ boost along $\vec{k}$

$m = 0$

standard vector $\bar{k}_\mu = \begin{pmatrix} \varkappa \\ 0 \\ 0 \\ \varkappa \end{pmatrix}$

$L^\mu{}_\nu k^\nu = k^\mu \Leftrightarrow \Sigma(\alpha_i J^i + \beta_i K^i)^\mu{}_\nu k^\nu = 0$

$$\begin{pmatrix} 0 & \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & 0 & -\alpha_3 & \alpha_2 \\ \beta_2 & \alpha_3 & 0 & -\alpha_1 \\ \beta_3 & -\alpha_2 & \alpha_1 & 0 \end{pmatrix} \begin{pmatrix} \varkappa \\ 0 \\ 0 \\ \varkappa \end{pmatrix} = \begin{pmatrix} \beta_3 \varkappa \\ (\beta_1 + \alpha_2)\varkappa \\ (\beta_2 - \alpha_1)\varkappa \\ \beta_3 \varkappa \end{pmatrix}$$

$\Rightarrow \beta_3 = 0$, $\beta_1 + \alpha_2 = 0$, $\beta_2 - \alpha_1 = 0$

$\ell = \alpha_3 J^3 + \beta_1 (K^1 - J^2) + \beta_2 (K^2 + J^1)$

Ex. 3. Write the element of the little group $L^\mu{}_\nu$.

$[J_3, K_1'] = iK_2'$

$[J_3, K_2'] = -iK_1'$

$[K_1', K_2'] = 0$

If we want finite-dim $\Rightarrow K_1'|\psi\rangle = K_2'|\psi\rangle = 0$

(see though 1302.1198, 1302.3225, ...
for ~~the theory of~~ more general reps)

$J_3|\psi\rangle = \lambda|\psi\rangle$, λ - half-integer

N.B. states $|\lambda\rangle$ and $|-\lambda\rangle$ are not related
unless we impose P

$U(L)|\lambda\rangle = e^{i\lambda\theta(L)}|\lambda\rangle$

$\theta(L_1 L_2) = \theta(L_1) + \theta(L_2)$

If $L = e^{i\alpha_3 J_3 + i\beta_1 K_1' + i\beta_2 K_2'} \Rightarrow \theta(L) = \alpha_3$

photons: $\lambda = \pm 1$

gravitons: $\lambda = \pm 2$

Constructing states with different momenta:

$$|\vec{k},\lambda\rangle \equiv U(\underbrace{\mathcal{L}^{\mu}_{\ \nu}(\vec{k})})|\lambda\rangle$$

$\uparrow$ definition of the state

standard transformation, that takes $\bar{k}^\mu \to k^\mu = (|\vec{k}|, \vec{k})$

(eg. boost along z-axis + rotation)

Definition of the group action:

$$U(\Lambda)|\vec{k},\lambda\rangle = U(\Lambda)U(\mathcal{L}(k))|\lambda\rangle =$$

$$= U(\mathcal{L}(\Lambda k))\, U(\underbrace{\mathcal{L}^{-1}(\Lambda k)\Lambda\mathcal{L}(k)}_{\bar{k}^\mu \to k^\mu \to (\Lambda k)^\mu \to \bar{k}^\mu})|\lambda\rangle$$

$\Rightarrow \mathcal{L}^{-1}(\Lambda k)\Lambda\mathcal{L}(k)$ belongs to the little group

$$U(\Lambda)|\vec{k},\lambda\rangle = U(\mathcal{L}(\Lambda k))\, e^{i\lambda\theta(\mathcal{L}^{-1}(\Lambda k)\Lambda\mathcal{L}(k))}|\lambda\rangle$$

$$\Rightarrow U(\Lambda)|\vec{k},\lambda\rangle \equiv e^{i\lambda\theta(\mathcal{L}^{-1}(\Lambda k)\Lambda\mathcal{L}(k))}|\Lambda k,\lambda\rangle \quad \circledast$$

This should be understood as a definition of the group action

N.B. Phase factor is important

Ex. 4 1) Check that the definition (*) satisfies the group properties

2) Check that the rep is unitary under the scalar product

$$\langle \vec{k}', \lambda' | \vec{k}, \lambda \rangle = \delta_{\lambda\lambda'} \, 2|k| \, \delta(\vec{k} - \vec{k}')$$

Creation operators:

$$a^+(\vec{k}, \lambda) |0\rangle = |\vec{k}, \lambda\rangle$$

$$U(\Lambda) \, a^+(k, \lambda) \, U^{-1}(\Lambda) = e^{i\lambda\theta(\mathcal{L}^{-1}(\Lambda k)\Lambda \mathcal{L}(k))} a^+(\Lambda k, \lambda)$$

$$U(\Lambda) \, a(k, \lambda) \, U^{-1}(\Lambda) = e^{+i\lambda\theta(\mathcal{L}^{-1}(k)\Lambda^{-1}\mathcal{L}(\Lambda k))} a(\Lambda k, \lambda)$$

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2) Properties of S-matrix with massless particles

$\lambda = \pm j$

$$\theta[\mathcal{L}^{-1}(k)\Lambda^{-1}\mathcal{L}(\Lambda k)]$$

$$S_\lambda(k, p) = e^{i\lambda\theta(k,\Lambda)} S_\lambda(\Lambda k, \Lambda p)$$

massless boson

$$\langle \ldots | \, a(k, \lambda) \ldots | 0 \ldots \rangle$$

Theorem

One can represent

$$S_{\pm j}(k,p) = \varepsilon_{\pm}^{*\mu_1}(k) \dots \varepsilon_{\pm}^{*\mu_j}(k)\, \underbrace{M_{\pm\mu_1\dots\mu_j}(\Lambda k,\Lambda p)}_{\text{symmetric tensor}}$$

$$\varepsilon_{\pm}^{\mu}(\hat{k}) = \underbrace{R(\hat{k})}_{\text{standard rotation from } z \text{ to } \hat{k}}{}^{\mu}{}_{\nu}\, \varepsilon_{\pm}^{\nu}$$

$$\varepsilon_{\pm}^{\nu} = \frac{1}{\sqrt{2}}(0,1,\pm i,0)$$ – circular polarization for a particle moving along z

Ex. 5 Prove relations

$$\varepsilon^{*}_{\pm\mu}(\hat{k})\, \varepsilon_{\pm}^{\mu}(\hat{k}) = 1 \;, \quad \varepsilon_{\pm\mu}(\hat{k})\, \varepsilon_{\pm}^{\mu}(\hat{k}) = 0$$

$$\varepsilon^{*\mu}_{\pm}(\hat{k}) = \varepsilon^{\mu}_{\mp}(\hat{k})$$

$$\varepsilon^{0}_{\pm}(k) = 0$$

$$k_{\mu}\, \varepsilon^{\mu}_{\pm}(k) = 0$$

$$\sum_{s=\pm} \varepsilon_{s}^{\mu}(\hat{k})\, \varepsilon_{s}^{*\nu}(\hat{k}) = \eta^{\mu\nu} + \frac{\bar{k}^{\mu}k^{\nu} + \bar{k}^{\nu}k^{\mu}}{2|\mathbf{k}|^{2}} \equiv \Pi^{\mu\nu}(\hat{k})$$

$$\sum_{s=\pm} \varepsilon_{s}^{\mu_1}(\hat{k})\, \varepsilon_{s}^{\mu_2}(\hat{k})\, \varepsilon_{s}^{*\nu_1}(\hat{k})\, \varepsilon_{s}^{*\nu_2}(k) = \frac{1}{2}\left(\Pi^{\mu_1\nu_1}\Pi^{\mu_2\nu_2} + \Pi^{\mu_1\nu_2}\Pi^{\mu_2\nu_1} - \Pi^{\mu_1\mu_2}\Pi^{\nu_1\nu_2}\right)$$

$\mathcal{E}^\mu_\pm$ are not vectors !

Ex. 6 Show that

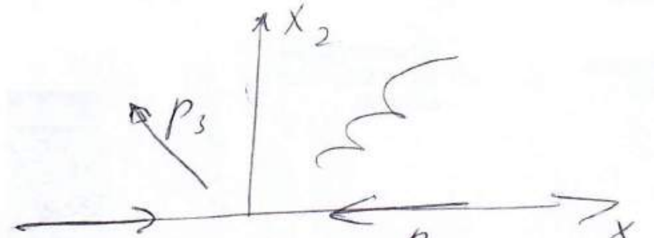
$$(\Lambda^{-1})^\mu{}_\nu \, \mathcal{E}^\nu_\pm(\Lambda \hat{k}) = e^{\pm i\theta(k,\Lambda)} \, \mathcal{E}^\mu_\pm(\hat{k}) + \frac{k^\mu}{|k|} (\Lambda^{-1})^0{}_\nu \, \mathcal{E}^\nu_\pm(\Lambda \hat{k})$$

Hints: a) act on $\mathcal{E}^\nu_\pm$ with the element $\mathcal{L}^{-1}(k)\Lambda^{-1}\mathcal{L}(\Lambda k)$ of the little group

b) use explicit form of a general element of the little group

c) argue $\mathcal{L}^\mu_\nu(k)\,\mathcal{E}^\nu_\pm = R^\mu{}_\nu(\hat{k})\,\mathcal{E}^\nu_\pm = \mathcal{E}^\mu_\pm(\hat{k})$

Proof of the theorem



choose a frame

Define in this frame

$$M_\pm^{\mu_1 \dots \mu_j}(k_c, p_c) \Leftarrow \mathcal{E}^{\mu_1}_\pm(\hat{k}_c) \dots \mathcal{E}^{\mu_j}_\pm(\hat{k}_c)\, S_{\pm j}(k_c, p_c)$$

Extend the definition to any frame by the laws of tensor transformations:

$$M_\pm^{\mu_1 \dots \mu_j}(k,p) = \underset{\substack{\uparrow \\ \text{takes } k_c, p_c \text{ to } k, p \;:\; \underline{k = \Lambda k_c}}}{\Lambda^{\mu_1}{}_{\nu_1}} \dots \Lambda^{\mu_j}{}_{\nu_j} \, M_\pm^{\nu_1 \dots \nu_j}(k_c, p_c)$$

Note that

$$k_{c\mu_1} M_\pm^{\mu_1\cdots\mu_j}(k_c,p_c)=0 \Rightarrow$$

$$\Rightarrow k_{\mu_1} M_\pm^{\mu_1\cdots\mu_j}(k,p)=0$$

Check

$$\varepsilon_\pm^{*\mu_1}(\hat{k})\cdots\varepsilon_\pm^{*\mu_j}(\hat{k})\, M^\pm_{\mu_1\cdots\mu_j}(k,p) = \qquad =(\Lambda^{-1})^{\nu_j}{}_{\mu_j}$$

$$= \varepsilon_\pm^{*\mu_1}(\hat{k})\cdots\varepsilon_\pm^{*\mu_j}(\hat{k})\,\Lambda_{\mu_1}{}^{\nu_1}\cdots\boxed{\Lambda_{\mu_j}{}^{\nu_j}}\,\varepsilon_{\pm\nu_1}(\hat{k}_c)\cdots\varepsilon_{\pm\nu_j}(\hat{k}).$$

$$\cdot\, S_{\pm j}(k_c,p_c) =$$

$$= \left(\varepsilon_\pm^{*\nu_1}(\Lambda^{-1}\hat{k})\, e^{\mp i\theta(\Lambda^{-1}k,\Lambda)} + \frac{(\Lambda^{-1}k)^{\nu_1}}{|\Lambda^{-1}k|}(\Lambda^{-1})^{0}{}_{\rho}\,\varepsilon_\pm^{\rho}(k)\right)$$

$$\cdots\; \varepsilon_{\pm\nu_1}(\hat{k}_c)\cdots\varepsilon_{\pm\nu_j}(\hat{k}_c)\, S_{\pm j}(k_c,p_c)$$

$$= \left(\varepsilon_\pm^{*\nu_1}(\hat{k}_c)\, e^{\mp i\theta(k_c,\Lambda)} + \frac{k_c^{\nu_1}}{|k_c|}(\Lambda^{-1})^{0}{}_{\rho}\,\varepsilon_\pm^{\rho}(k)\right)$$

$$\cdots\; \varepsilon_{\pm\nu_1}(\hat{k}_c)\cdots\varepsilon_{\pm\nu_j}(\hat{k}_c)\, S_{\pm j}(k_c,p_c).$$

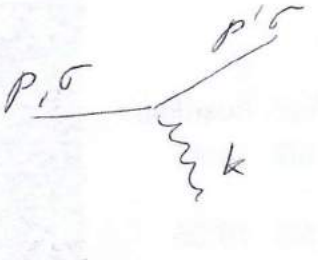
$$= e^{\mp ij\theta(k_c,\Lambda)}\, S_{\pm j}(k_c,p_c) =$$

$$= e^{\mp ij\theta(k_c,\Lambda)}\, e^{ij\theta(k_c,\Lambda)}\, S_{\pm j}(\Lambda k_c,\Lambda p_c) =$$

$$= S_{\pm j}(k,p)$$

$\Rightarrow$ Q.E.D.

3) Definition of charge and graviton couplings

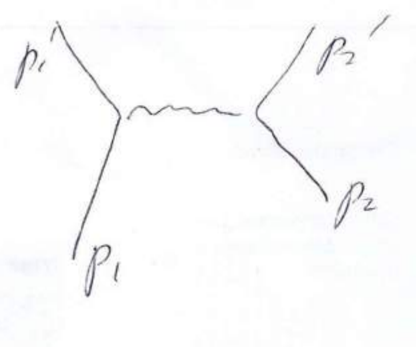


$k \ll p$

$$\varepsilon^{*\mu_1}_{\pm}(k) \dots \varepsilon^{*\mu_j}_{\pm}(k)\, p_{\mu_1} \dots p_{\mu_j} \underbrace{\delta_{\sigma\sigma'}}_{\text{try diag in polarizations}}$$

photon : $$\frac{2ie\, \varepsilon^{*\mu}_{\pm}(k)\, p_\mu}{(2\pi)^{3/2}}\, \delta_{\sigma\sigma'}\, (2\pi)^4$$

graviton $$\frac{2if}{M_p^2}\, \frac{\left(p_\mu\, \varepsilon^{*\mu}_{\pm}(k)\right)^2}{(2\pi)^{3/2}}\, \delta_{\sigma\sigma'}\, (2\pi)^4$$



$$S = \sum_{s=\pm} - \frac{4 e_1 e_2\, p_{1\mu}\, p_{2\nu}\, \varepsilon^{*\mu}_{s}(k)\, \varepsilon^{\nu}_{s}(k)}{k^2} \left(\frac{-i}{(2\pi)^2}\right)$$

$$= -i\, \frac{e_1 e_2}{\pi^2 t}\, p_{1\mu}\, p_{2\nu} \sum_{s=\pm} \varepsilon^{*\mu}_{s}(k)\, \varepsilon^{\nu}_{s}(k)$$

$\nwarrow$ $t = -(p_1 - p_1')^2$

<u>Ex. 7</u> Derive :

$$S = -\frac{i}{\pi^2 t}\left[e_1 e_2 \, (p_1 \cdot p_2) + \frac{f_1 f_2}{M_p^2}\left((p_1 p_2)^2 - \frac{1}{2} m_1^{\,2} m_2^{\,2}\right)\right] + O(1)$$

Compare with the amplitude of scattering in a Coulomb field and in a linearized Schwartzschild metric

<u>Assume</u> $e_+ = e_-$

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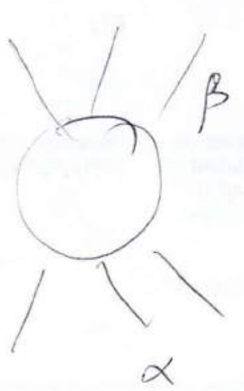
4) Charge conservation and equivalence principle



$= S_{\beta\alpha}$

$\sum_n$

$= S^{\lambda}_{\beta\lambda}$ $\quad \lambda = \pm 1, \pm 2, ..$

out $\dfrac{1}{(p_n+k)^2 + m^2} = \dfrac{1}{2 p_n \cdot k}$

in: $\dfrac{1}{(p_n-k)^2 + m^2} = -\dfrac{1}{2 p_n \cdot k}$

$$S^{\pm 1}_{\beta\alpha} = \sum_n \frac{\eta_n}{2p_n \cdot k} \cdot \frac{\left(\varepsilon^{*\mu}_{\pm}(\hat{q})\, p^{(n)}_{\mu}\, 2 i e_n\right)(2\pi)^4}{(2\pi)^{3/2}} \left(\frac{-i}{(2\pi)^4}\right) S_{\beta\alpha}$$

$$= \frac{S_{\beta\alpha}}{(2\pi)^{3/2}} \sum_n e_n \underset{\uparrow}{\eta_n} \frac{\varepsilon^{*\mu}(\hat{k})\, p_{n\mu}}{p_n \cdot k}$$

$$\eta_n = \begin{cases} +1 & \text{outgoing} \\ -1 & \text{incoming} \end{cases}$$

$$M_\mu = \frac{S_{\beta\alpha}}{(2\pi)^{3/2}} \sum_n e_n \eta_n \frac{p_{n\mu}}{p_n \cdot k}$$

$$0 = k_\mu \cdot M_\mu = \frac{S_{\beta\alpha}}{(2\pi)^{3/2}} \sum_n e_n \eta_n \Rightarrow \sum_n e_n \eta_n = \underline{\sum_{out} e_n - \sum_{in} e_n = 0}$$

$$S^{\pm 2}_{\beta\alpha} = \sum_n \frac{\eta_n}{2p_n \cdot k} \frac{\left(\varepsilon^{*\mu}_{\pm}(\hat{k})\, p_{n\mu}\right)^2 2 i f_n \cdot (2\pi)^4}{(2\pi)^{3/2} M_p} \frac{-i}{(2\pi)^4} S_{\beta\alpha}$$

$$\Rightarrow M_{\mu_1\mu_2} = \frac{S_{\beta\alpha}}{(2\pi)^{3/2} M_p} \sum_n f_n \eta_n \frac{p_{n\mu_1} p_{n\mu_2}}{p_n \cdot k}$$

$$\Rightarrow k_{\mu_1} M_{\mu_1\mu_2} \propto \sum_n f_n \eta_n p_{n\mu_2} = 0$$

Solution: all f_n are equal.

Then $\sum \eta_n p_{n\mu} = 0$ due to the conservation of energy and momentum

N.B. 1) $f_+ \neq f_-$ in general, if we don't impose parity

2) Any particles in hard lines, including gravitons!

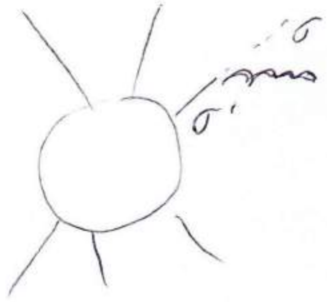
5) Higher spins?

$$\sum g_n^{(j)} \eta_n p_{n\mu_1} \sim p_{n\mu_j} = 0 \quad , \quad j>2$$

If no restrictions on kinematics $\Rightarrow g_n^{(j)} = 0$

Ex. 8 Does this argument exclude any interaction of massless higher spins?

Other couplings?



$$\sum_n S_{\beta\alpha}^{\text{spin-flipped}} \, \bar{e}_n \eta_n = 0$$

Impossible to cancel