Lecture 2

1) Recap. from lecture 1

a) $U(\Lambda)\,|\vec{k},\lambda\rangle = e^{i\lambda\theta\left(\underbrace{L^{-1}(\Lambda k)\,\Lambda\, L(k)}_{\text{element of the little group}}\right)}\,|\Lambda\vec{k},\lambda\rangle$

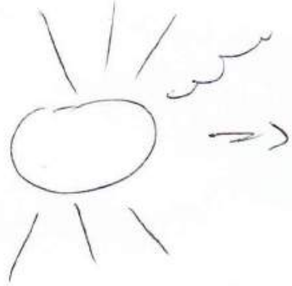
$L(k)$ - standard transformation

$$\bar{k}^\mu \xrightarrow{L} k^\mu = (|\vec{k}|, \vec{k})$$

$(\varkappa, 0, 0, \varkappa)$

$$\langle k,\lambda|k',\lambda'\rangle = 2|\vec{k}|\,\delta(\vec{k}-\vec{k}')\,\delta_{\lambda\lambda'}$$

b) $\lambda = \pm j$, j - integer



$$\Rightarrow \quad S_{\pm j}(k,p) = \varepsilon_{\pm}^{*\mu_1}(\vec{k}) \dots \varepsilon_{\pm}^{*\mu_j}(\vec{k})\, M_{\pm\mu_1\dots\mu_j}$$

$$k_{\mu_1} M_{\mu_1\dots\mu_j} = 0$$

$$\varepsilon_{\pm}^{\mu}(\vec{k}) = R(k)^{\mu}{}_{\nu}\, \bar{\varepsilon}_{\pm}^{\nu}$$

$$\bar{\varepsilon}_{\pm}^{\nu} = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$$

c) ②

$k \ll p$

$$\propto \varepsilon_{\pm}^{*\mu_1}(\hat{k}) \cdots \varepsilon_{\pm}^{*\mu_j}(\hat{k})\, p_{\mu_1} \cdots p_{\mu_j}\, \delta_{\sigma\sigma'}$$

N.B. Not necessarily diagonal in polarization σ, σ' at this stage, but see later

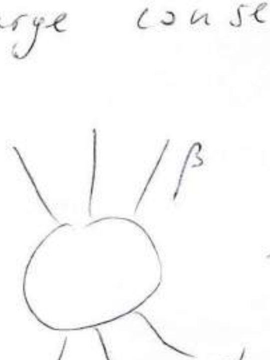
photon :

$$\frac{2 i e_{\pm}\, \varepsilon_{\pm}^{*\mu}(\hat{k})\, p_{\mu}}{(2\pi)^{3/2}}\, \delta_{\sigma\sigma'}\, (2\pi)^4$$

graviton :

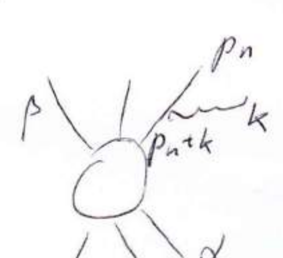
$$\frac{2i f_{\pm}}{M_p}\, \frac{\left(p_{\mu}\, \varepsilon_{\pm}^{*\mu}(\hat{k})\right)^2}{(2\pi)^{3/2}}\, \delta_{\sigma\sigma'}\, (2\pi)^4$$

2) Charge conservation



$= S_{\beta\alpha}$

$\sum_n$



$= S^{\lambda}_{\beta\alpha}$

out : $\dfrac{1}{(p_n+k)^2+m^2} \approx \dfrac{1}{2(p_n \cdot k)}$

in : $\dfrac{1}{(p_n-k)^2+m_n^2} \approx -\dfrac{1}{2(p_n \cdot k)}$

$$S^{\pm 1}_{\beta\alpha} = \sum_n \frac{\eta_n}{2(p_n \cdot k)} \frac{(\varepsilon^{*\mu}_{\pm} \cdot p_{\mu n}) \cdot 2 i e_n (2\pi)^4}{(2\pi)^{3/2}} \left(-\frac{i}{(2\pi)^4}\right) S_{\beta\alpha}$$

$$\eta_n = \begin{cases} +1 & \text{- outgoing} \\ -1 & \text{- ingoing} \end{cases}$$

$$S^{\pm 1}_{\beta\alpha} = \frac{S_{\beta\alpha}}{(2\pi)^{3/2}} \sum_n e_n \eta_n \frac{\varepsilon^{*\mu}_{\pm}(\hat{k}) p_{n\mu}}{p_n \cdot k}$$

$$\varepsilon^{*\mu} \to k^\mu \Rightarrow 0 = k_\mu M^\mu \propto \sum_n e_n \eta_n$$

$$\Rightarrow \sum e_n \eta_n = \sum_{out} e_n - \sum_{in} e_n = 0$$

3) Equivalence principle

$$S^{\pm 2}_{\beta\alpha} = \frac{S_{\beta\alpha}}{(2\pi)^{3/2}} \sum_n \frac{f_n}{M_p} \eta_n \frac{(\varepsilon^{*\mu}_{\pm}(\hat{k}) p_{n\mu})^2}{(p_n \cdot k)}$$

$$0 = \left(\sum_n \eta_n f_n p_{n\mu}\right) \varepsilon^{*\mu}_{\pm} S_{\beta\alpha} + \text{no additional kinematic constraints}$$

$\Rightarrow$ all f_n are equal $\Rightarrow \sum \eta_n p_{n\mu} = 0$ due to energy – momentum conservation

N.B. a) $f_+ \neq f_-$ in general (unless parity)

b) graviton can be also in the hard lines

$\Rightarrow$ ~~interact~~ self interaction of gravitons

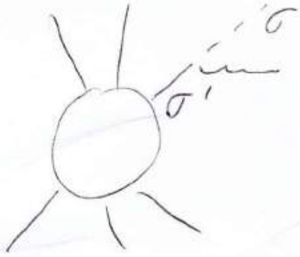
4) Higher spins?

$$\sum_n g_n^{(j)} \eta_n p_{n\mu_1} \cdots p_{n\mu_{j-1}} = 0$$

If no restrictions on kinematics $\Rightarrow g_n^{(j)} = 0$

Ex. 8 Does this argument exclude any interactions of massless higher spins?

Other couplings?



$$\sum_n S_{\beta\alpha}^{\text{spin-flipped}} \bar{e}_n \eta_n = 0$$

But spin-flipped amplitudes ar different functions of the hard momenta

$\Rightarrow$ cancellation is impossible

B. Maxwell equations

5) Free massless fields

creation operators

$$a^+(\vec{k}, \lambda)\,|0\rangle = |\vec{k}, \lambda\rangle$$

$$U(\Lambda)\, a^+(k, \lambda)\, U^{-1}(\Lambda) = e^{i\lambda\theta}\, a^+(\Lambda k, \lambda)$$

$$U(\Lambda)\, a(k, \lambda)\, U^{-1}(\Lambda) = e^{-i\lambda\theta}\, a(\Lambda k, \lambda)$$

definition of a local field $\psi_n(x)$

1. $\psi_n(x)$ - linear combination of a, a^+
2. $i\,[P_\mu, \psi_n(x)] = \partial_\mu \psi_n(x)$
3. $\psi_n(x)$ transforms in a ~~finite~~ finite-dimensional irrep of the Lorentz group

$$U(\Lambda)\, \psi_n(x)\, U^{-1}(\Lambda) = \sum_m D_{nm}(\Lambda^{-1})\, \psi_m(\Lambda x)$$

4. Locality: $[\psi_n(x), \psi_m(y)] = 0$ outside the light-cone

Needed to have Lorentz ~~covariant~~ invariant time-ordering

First try: annihilation field

$$\psi_n(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2|k|} e^{ikx} a(\vec{k}, \lambda) w_n(\vec{k}, \lambda)$$

$$e^{-i\lambda\theta(\mathcal{L}^{-1}(\Lambda k)\Lambda \mathcal{L}(k))} w_n(\vec{k}, \lambda) = \sum_m D_{nm}(\Lambda^{-1}) w_m(\vec{k}, \lambda)$$

Take $k_\mu = \bar{k}_\mu$

$$w_m(\bar{k}, \lambda) \equiv \bar{w}_m(\lambda)$$

$$\Rightarrow e^{-i\lambda\theta(\mathcal{L}^{-1}(\Lambda\bar{k})\,;\,\Lambda)} \bar{w}_n = \sum_m D_{nm}(\Lambda^{-1}) \bar{w}_m$$

Let $\Lambda = L^{-1}$ belong to the little group

$$e^{+i\lambda\theta(L)} \bar{w}_n = \sum_m D_{nm}(L) \bar{w}_m$$

$$L = 1 + i\alpha_3 J_3 + i\beta_1 (K_1 - J_2) + i\beta_2 (K_2 + J_2)$$

$$\theta = \alpha_3$$

$$\Rightarrow \quad D_{nm}(J_3) \bar{w}_m = \lambda \bar{w}_n$$

$$D_{nm}(K_1 - J_2) \bar{w}_m = D_{nm}(K_2 + J_1) \bar{w}_m = 0$$

$$J_i = N_i^+ + N_i^- , \quad K_i = -i(N_i^+ - N_i^-).$$

independent $SU(2)$ generators

$$D_{nm}(N_1^+ - iN_2^+)\bar{w}_m = D_{nm}(N_1^+ + iN_2^+)\bar{w}_m = 0$$

$$\Rightarrow D(N_3^+)\bar{w} = j_+ \bar{w} \quad ; \quad D(N_3^-) w = -j_- \bar{w}$$

$$j_+ - j_- = \lambda$$

$\Rightarrow$ $\bar{w}_n$ lies in the $(j_- + \lambda, j_-)$ representation

Basic rep. $(\lambda, 0)$. The rest come from derivatives

Examples!

photons $\lambda = \pm 1 \Rightarrow \psi_n$ in $(1,0)$ or $(0,1)$

$F^+_{\mu\nu}$ $\quad F^-_{\mu\nu}$

no $A_\mu \in (\frac{1}{2}, \frac{1}{2})$!

gravitons $\lambda = \pm 2 \Rightarrow \underbrace{(2,0) \text{ or } (0,2)}_{\text{Riemann tensor}}$

no $g_{\mu\nu} \in (1,1)$!

Explicitly for photons

$$F_+^{\mu\nu} = i\int \frac{d^3k\, e^{ikx}}{(2\pi)^{3/2}\, 2k}\left(k^\mu \varepsilon_+^\nu(\vec{k}) - k^\nu \varepsilon_+^\mu(\vec{k})\right) a(k,+)$$

Ex. 9. Check that this field transforms as a $(1,0)$ tensor

But this does not satisfy locality!

$$\left[F_+^{\mu\nu}(x),\ \left(F_+^{\lambda\rho}(y)\right)^* \right]$$

$$\propto \int \frac{d^3k}{2|k|}\left(k^\mu \varepsilon_+^\nu - k^\nu \varepsilon_+^\mu\right)\left(k^\lambda \varepsilon_+^{*\rho} - k^\rho \varepsilon_+^{*\lambda}\right) e^{ik(x-y)}$$

Take $x^0 = y^0$, $\mu=\lambda=0$, $\nu=\rho=i$

$$\Rightarrow \propto \int \frac{d^3k}{2|k|}\, e^{i\vec{k}(\vec{x}-\vec{y})}\, |k|^2\, \varepsilon_+^i\, \varepsilon_+^{*i} = \underbrace{\int \frac{d^3k\, |k|}{2}\, e^{i\vec{k}(\vec{x}-\vec{y})}}_{\neq 0}$$

We need opposite helicity

$a^+(k,-)$ transforms in the same way as $a(k,+)$

$$\Rightarrow F_+^{\mu\nu} = i\int \frac{d^3k}{(2\pi)^{3/2}\, 2|k|}\left(k^\mu \varepsilon_+^\nu - k^\nu \varepsilon_+^\mu\right)\left(a(k,+)e^{ikx} + \eta\, a^+(k,-)e^{-ikx}\right)$$

Ex. 10 Show that $F_+^{\mu\nu}$ commutes with $(F_+^{\lambda\rho})^+$ at space-like separations if $|\eta|^2 = 1$

We choose $\eta = -1$

N.B. $(F_+^{\mu\nu})^+ = F_-^{\mu\nu}$

6) Interaction

Attempt to use $F^{\pm}_{\mu\nu}$

But $\langle T\left(F^{\pm}_{\mu\nu}(k)\, F^{\pm}_{\lambda\rho}(-k)\right)\rangle \propto \dfrac{k_\mu k_\lambda g_{\nu\rho}+\ldots}{k^2}$

$\Rightarrow$ no pole at $k^2 \to 0 \Rightarrow$ no long-range force

Notice $F^{\mu\nu}_{\pm} = \partial_\mu A_{\pm\nu} - \partial_\nu A_{\pm\mu}$

$$A^\mu_{\pm} = \int \frac{d^3k}{(2\pi)^{3/2}\, 2k}\, \varepsilon^\mu_{\pm}(\vec{k})\left(a(k,+)e^{ikx} + a^+(k,-)e^{-ikx}\right)$$

$A^0_{\pm} = 0 \Rightarrow A^\mu_{\pm}$ are <u>not</u> vectors

Indeed, we have seen that it is impossible to have $(1/2, 1/2)$ field for photons

<u>Ex. 11</u> Show that

$$U(\Lambda)\, A^\mu_{\pm}(x)\, U(\Lambda)^{-1} = \Lambda^\mu_{\ \nu}\, A^\nu_{\pm}(\Lambda x) + \partial_\mu \varphi_{\pm}(x,\Lambda)$$

Real combinations

$$\left.\begin{aligned} A_\mu &= A_{+\mu} + A_{-\mu} && \text{– vector} \\ B_\mu &= -i\,(A_{+\mu} - A_{-\mu}) && \text{– pseudo-vector} \end{aligned}\right\} \text{ under } P \text{ and } T$$

Focus on A_μ

$$\Delta_{\mu\nu}(k) = \langle T(A_\mu(k) A_\nu(-k)\rangle =$$

$$= -i \sum_{s=\pm} \frac{(\varepsilon_s^\mu \varepsilon_s^{*\nu})(\vec{k})}{k^2 - i\varepsilon} = -\frac{i}{k^2 - i\varepsilon} \underbrace{\left(\eta_{\mu\nu} + \frac{k_1^\mu k_2^\nu + k_1^\nu k_2^\mu}{2|\vec{k}|^2}\right)}_{\Pi^{\mu\nu} \text{ from lecture 1}}$$

$k_1^\nu = (|\vec{k}|, \vec{k})$

$k_2^\nu = (|\vec{k}|, -\vec{k})$

$$\Pi^{\mu\nu} = \eta^{\mu\nu} + \underbrace{(n^\mu k^\nu + n^\nu k^\mu)\frac{k^0}{|k|^2} - \frac{k^\mu k^\nu}{|k|^2}}_{\text{grad}} + \underbrace{\frac{k^2 n^\mu n^\nu}{|k|^2}}_{\text{time-like}}$$

$n^\mu = (1, 0, 0, 0)$

$$\Delta_{\mu\nu}(k) = \underbrace{\frac{\eta^{\mu\nu}}{k^2}}_{\text{Lorent covariant}} + \underbrace{\frac{1}{k^2}(k^\mu \tilde{n}^\mu + \tilde{n}^\nu k^\mu)}_{\Delta^{\mu\nu}_{grad}} + \underbrace{\frac{n^\mu n^\nu}{|k|^2}}_{\text{no phisical pole} \Leftrightarrow \text{instantaneous interaction } \Delta^{\mu\nu}_{inst}}$$

$$H_{int} = \int d^3x \, (-A_i J^i + \dots)$$

~~Δ_{grad}~~ Non-covariant contributions must cancel

$\Delta^{\mu\nu}_{grad}$ vanishes if J^μ is conserved, $\partial_\mu J^\mu = 0$

$\Delta^{\mu\nu}_{inst}$ cancelled by an explicit non-local term:

$$H_{int} = \int d^3x \, (-A_i J^i) + \frac{1}{2} \int d^3x \, d^3y \, J^0(\vec{x}; t) D(\vec{x} - \vec{y}) J^0(\vec{y}; t)$$

$$D(\vec{x}) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{|k|^2} e^{ikx} := \frac{1}{4\pi |\vec{x}-\vec{y}|}$$

Convenient to introduce $A_0(x,t)$ with the propagator

$$\langle A_0(x,t), A_0(x',t') \rangle = \frac{\delta(t-t')}{4\pi |x-x'|}$$

$\Rightarrow$ Recover QED in the Coulomb gauge

Ex. 12 Assume that the theory also contains other fields charged under B_μ. What is the physical interpretation of such interaction?

Compute the propagator

$$\langle T(A_\mu(x)\, B_\nu(-k)) \rangle$$

Is it possible to decompose it into Lorentz-covariant, gradient and instantaneous parts?

Analyse physical implications of the result.