

Lecture 3

①

1) Derivation of the Maxwell equations

Free field

$$A_{\text{free}}^i(x) = \sum_{s=\pm} \int \frac{d^3k}{(2\pi)^{3/2} 2k} \epsilon_s^i(k) (a(k,s)e^{ikx} + a^\dagger(k,-s)e^{-ikx})$$

$$\Box A_{\text{free}}^i = 0$$

$$\partial_i A_f^i = 0$$

$$[A_f^i(\vec{x}), A_f^j(\vec{y})] = [\dot{A}_f^i(\vec{x}), \dot{A}_f^j(\vec{y})] = 0$$

$$[A_f^i(\vec{x}), \dot{A}_f^j(\vec{y})] = i(\delta_{ij}\delta(\vec{x}-\vec{y}) + \partial_i \partial_j D(\vec{x}-\vec{y}))$$

$$\frac{1}{4\pi|\vec{x}-\vec{y}|} = \int \frac{d^3k}{(2\pi)^3 |\vec{k}|^2} e^{i\vec{k}(\vec{x}-\vec{y})}$$

Ex. 13 Prove these commutation relations

Note: $[A, \dot{A}]$ does not vanish at space-like separations $\Rightarrow A_\mu$ is not a local field

$$\text{Hint} = - \int d^3x A_i J^i + \frac{1}{2} \int d^3x d^3y J^0(\vec{x}) D(\vec{x}-\vec{y}) J^0(\vec{y})$$

$$\partial_\mu J^\mu = 0 \quad - \text{conserved}$$

equal-time commutators

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$$\ddot{A}_i = i [H, \dot{A}_i] = i [H_f, \dot{A}_i] + i [H_{int}, \dot{A}_i] =$$

$$= \partial_j^2 A_i - i \int d^3 y [A_j(\vec{y}) J^j(\vec{y}), \dot{A}_i(x)]$$

If J commutes with A (true for QED)

$$\Rightarrow \ddot{A}_i = \partial_j^2 A_i + \int d^3 y J^j(y) (\delta_{ij} \delta(y-x) + \partial_i \partial_j D(x-y))$$

$$= \partial_j^2 A_i + J_i(x) + \partial_i \partial_j \int d^3 y D(y-x) J^j(y)$$

$$\Rightarrow \square A_i = -J_i(x) + \partial_i \partial_j \int d^3 y D(x-y) J^j(y) =$$

$$= -J_i(x) + \partial_i \partial_0 \int d^3 y D(x-y) J^0(y)$$

Introduce an artificial operator:

$$A_0(\vec{x}, t) = \int d^3 y D(\vec{x} - \vec{y}) J_0^{\text{eff}}(\vec{y}, t)$$

Makes sense:

$$\delta S_{\beta\alpha} = -i \langle \beta | \int dt H_{int} | \alpha \rangle =$$

$$= i \langle \beta | \int d^4 x A_i | \alpha \rangle \delta J^i(x) -$$

$$- i \langle \beta | \int dt \underbrace{\int d^3 y J^0(y, t) D(x-y)}_{A^0(x, t)} | \alpha \rangle \delta J^0(x)$$

$$= i \langle \beta | \int d^4 x A_F(x) | \alpha \rangle \delta J^F(x)$$

$$\partial_j^2 A_0 = -J_0 \Rightarrow \partial_j (\partial_j A_0 - \partial_0 A_j) = -J_0$$

(3)

$$\square A_i = -J_i + \partial_i \partial_0 A^0 \Rightarrow -\partial_0 (\partial_0 A_i - \partial_i A_0) + \partial_j (\partial_j A_i - \partial_i A_j) = -J_i$$

$$\boxed{\partial^\mu F_{\mu\nu} = -J_\nu}$$

C. Derivation of (linearized) Einstein eqs.

2) Free fields

$$R_S^{\mu\nu\lambda\rho} = - \int \frac{d^3 k}{(2\pi)^{3/2} 2k} \cancel{e^{ikx}} (k^\mu \varepsilon_S^\nu - k^\nu \varepsilon_S^\mu) \cdot$$

$$\cdot (k^\lambda \varepsilon_S^\rho - k^\rho \varepsilon_S^\lambda) (a(k, s) e^{ikx} + a^\dagger(k, -s) e^{-ikx})$$

Ex. 14 Show that this is (2,0) or (0,2) tensor

$$h_S^{ij} = \int \frac{d^3 k}{(2\pi)^{3/2} 2k} \cancel{e^{ikx}} \varepsilon_S^i(k) \varepsilon_S^j(k) (a(k, s) e^{ikx} + a^\dagger(k, -s) e^{-ikx})$$

$$h_S^{0i} = h_S^{i0} = h_S^{00} = 0 ; \quad \partial_i h^{ij} = h^i_i = 0$$

$$R_S^{\mu\nu\lambda\rho} = \partial^\lambda \partial^\nu h_S^{\mu\rho} - \partial^\lambda \partial^\rho h_S^{\mu\nu} - \partial^\nu \partial^\lambda h_S^{\mu\rho} + \partial^\nu \partial^\rho h_S^{\mu\lambda}$$

cf. the linearized Riemann tensor

Hence forth we use $h_{\mu\nu} = h_{\mu\nu+} + h_{\mu\nu-}$

(4)

~ Interactions are P and T even

Comment on more general couplings at the end

Ex 15 compute equal-time commutators

$$[h^{ij}(\vec{x}), h^{kl}(\vec{y})] = [\dot{h}^{ij}(\vec{x}), \dot{h}^{kl}(\vec{y})] = 0$$

$$[h^{ij}(\vec{x}), \dot{h}^{kl}(\vec{y})] = \frac{i}{2} (\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} - \delta^{ij} \delta^{kl}) \delta(\vec{x} - \vec{y})$$

$$+ \frac{i}{2} (\partial^i \partial^k \delta^{jl} + \partial^j \partial^l \delta^{ik} + \partial^i \partial^l \delta^{jk} + \partial^j \partial^k \delta^{il} - \\ - \partial^i \partial^j \delta^{kl} - \partial^k \partial^l \delta^{ij}) D(\vec{x} - \vec{y}) +$$

$$+ \frac{i}{2} \partial^i \partial^j \partial^k \partial^l \delta(\vec{x} - \vec{y})$$

$$\delta(\vec{x} - \vec{y}) = - \frac{|\vec{x} - \vec{y}|}{8\pi} = \int \frac{d^3k}{(2\pi)^3 |k|^4} e^{i\vec{k}(\vec{x} - \vec{y})}$$

3) Interactions

$$\Delta_{\mu\nu\lambda\rho}(k) \equiv \langle T(h_{\mu\nu}(k) h_{\lambda\rho}(-k)) \rangle = -\frac{i}{k^2 - i\epsilon} \Pi_{\mu\nu\lambda\rho}(k)$$

$$\Pi^{\mu\nu\lambda\rho}(k) = \sum_s \epsilon_s^\mu \epsilon_s^\nu \epsilon_s^{*\lambda} \epsilon_s^{*\rho} =$$

$$= \frac{1}{2} (\Pi^{\mu\lambda} \Pi^{\nu\rho} + \Pi^{\mu\rho} \Pi^{\nu\lambda} - \Pi^{\mu\nu} \Pi^{\lambda\rho})$$

$$\sum_s \epsilon_s^\mu \epsilon_s^{*\nu} \rightarrow \Pi^{\mu\nu} = \left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{|k|^2} + \frac{k^0 (\eta^{\mu\lambda} k^\nu + \eta^{\nu\lambda} k^\mu)}{|k|^2} + \frac{k^2 \eta^{\mu\lambda} \eta^{\nu\rho}}{|k|^2} \right)$$

$$\eta^{\mu\lambda} = (1, 0, 0, 0)$$

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$$H_{\text{int}} = - \int d^3x \, h_{ij}(x) T^{ij}(x) + \dots$$

$$\Delta_{\mu\nu\lambda\rho}^{\text{cov}} = \frac{1}{2} (g_{\mu\lambda} g_{\nu\rho} + g_{\mu\rho} g_{\nu\lambda} - g_{\mu\nu} g_{\lambda\rho}) \left(\frac{-i}{k^2 - i\varepsilon} \right)$$

$$\Delta_{\mu\nu\lambda\rho}^{\text{grad}} = k^\mu () + k^\nu () + k^\lambda () + k^\rho ()$$

$$\Delta_{\mu\nu\lambda\rho}^{\text{inst}} = \frac{1}{2} \left(\overbrace{g_{\mu\lambda} n_\nu n_\rho + g_{\mu\rho} n_\nu n_\lambda + g_{\nu\lambda} n_\mu n_\rho + g_{\nu\rho} n_\mu n_\lambda}^{P_{\mu\nu\lambda\rho}^{(1)}} - g_{\mu\nu} n_\lambda n_\rho - g_{\lambda\rho} n_\mu n_\nu \right) \frac{1}{|k|^2} +$$

$$+ \frac{k^2 \overbrace{n_\mu n_\nu n_\lambda n_\rho}^{P_{\mu\nu\lambda\rho}^{(2)}}}{2 |k|^4}$$

$$\frac{1}{(2\pi)^4} i \int d^4k \, e^{ik(x-y)} \Delta_{\mu\nu\lambda\rho}^{\text{inst}} = \frac{1}{2} (P_{\mu\nu\lambda\rho}^{(1)} + P_{\mu\nu\lambda\rho}^{(2)}) \delta(x^0 - y^0) D(\vec{x} - \vec{y})$$

$$+ \frac{1}{2} P_{\mu\nu\lambda\rho}^{(2)} \ddot{\delta}(x^0 - y^0) \varepsilon(\vec{x} - \vec{y})$$

$$\Rightarrow H_{\text{int}} = - \int d^3x \, h_{ij}(x) T^{ij}(x) +$$

$$+ \frac{1}{2} \int d^3x \, d^3y \left[2 T_{00}^\wedge(x, t) T_{\mu 0}(y, t) - \right.$$

$$- \frac{1}{2} T_\mu^\wedge(x, t) T_{00}(y, t) - \frac{1}{2} T_{00}(x, t) T_\mu^\wedge(y, t)$$

$$+ \frac{1}{2} T_{00}(x, t) T_{00}(y, t) \left. \right] D(x-y) -$$

$$- \frac{1}{4} \int d^3x \, d^3y \, \ddot{T}_{00}(x, t) \varepsilon(x-y) \dot{T}_{00}(y, t)$$

$\partial_\mu T_{\mu\nu} = 0 \Rightarrow T_{\mu\nu}$ is the energy-momentum tensor

4) Einstein equations

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$$\square h_{ij} = i \left[\int d^3y h_{\mu\nu}(y) \overset{\uparrow}{T^{\mu\nu}}(y), h_{ij}(x) \right]$$

The can be taken out of the commutator only at the leading order

\Rightarrow only linearized gravity

$$\square h_{ij} = -T_{ij} + \frac{1}{2} \delta_{ij} T^{\mu\mu}$$

$$= \int d^3y \left(T^{jk}(y) \partial_i \partial_k + T^{ik}(y) \partial_j \partial_k - \frac{1}{2} T_k^k \partial_i \partial_j - \frac{1}{2} T^{\mu\mu} \delta_{ij} \partial_k \partial_k \right) D(x-y) -$$

$$- \frac{1}{2} \partial_i \partial_j \partial_k \partial_k \int d^3y T^{\mu\mu}(y) \varepsilon(x-y)$$

Invent artificial components, so that

$$\delta S_{\beta\alpha} = -i \int dt \langle \beta | \delta H_{int} | \alpha \rangle = i \int d^4x \langle \beta | \hat{h}_{\mu\nu} | \alpha \rangle \delta T^{\mu\nu}$$

$$\left\{ \begin{aligned} \hat{h}_{00}(x) &= \frac{1}{2} \int d^3y D(x-y) (T_{00}(y) + T_i^i(y)) - \frac{1}{2} \int d^3y \varepsilon(x-y) \ddot{T}_{00}(y) \\ \hat{h}_{0i}(x) &= \int d^3y D(x-y) T_{0i}(y) \\ \hat{h}_{ij}(x) &= h_{ij}(x) + \underbrace{\frac{\delta_{ij}}{2} \int d^3y D(x-y) T_{00}(y)}_{\text{added trace}} \end{aligned} \right.$$

equations take local form:

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$$\partial_j^2 \hat{h}_{00} = -\frac{1}{2} (T_{00} + T_i^i) + \frac{1}{3} \hat{h}^{ii}$$

$$\partial_j^2 \hat{h}_{0i} = -T_{0i}$$

$$\square \hat{h}^{ij} = -T^{ij} + \frac{1}{2} \delta^{ij} T_\mu^\mu + \partial_0 \partial_i \hat{h}^{0j} + \partial_0 \partial_j \hat{h}^{0i} + \partial_i \partial_j \hat{h}_{00} - \partial_i \partial_j \frac{1}{3} \hat{h}^k_k$$

plus "gauge-fixing" conditions"

$$\begin{cases} \partial_i \hat{h}^{ij} = \frac{1}{3} \partial_j \hat{h}^k_k \\ \partial_i \hat{h}^{i0} = -\frac{2}{3} \partial_0 \hat{h}^k_k \end{cases}$$

Ex 16 Check that these equations are equivalent to:

$$\square \hat{h}_{\mu\nu} - \partial_\mu \partial_\lambda \hat{h}^\lambda_{\nu} - \partial_\nu \partial_\lambda \hat{h}^\lambda_{\mu} + \partial_\mu \partial_\nu \hat{h}^\lambda_\lambda =$$

$$\nearrow = -T_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} T^\lambda_\lambda$$

linearized Ricci

Now we see the linearized gauge invariance

$$\hat{h}_{\mu\nu} \mapsto h_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$$

7) A trick to cut the iterations short

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a) work with $H_{\mu\nu} = -h_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} h^\lambda_\lambda$

(linear expansion of $\sqrt{-g} g^{\mu\nu}$)

b) introduce an auxiliary field $\Gamma^\lambda_{(\mu\nu)}$

Lorentz tensor,
symmetric in $\mu\nu$.

(linearized Christoffels)

$$S^{(2)} = \int d^4x \left[H^{\mu\nu} (\partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma_\mu) + \eta^{\mu\nu} (\Gamma^\lambda_{\mu\nu} \Gamma_\lambda - \Gamma^\lambda_{\mu\lambda} \Gamma^\rho_{\nu\rho}) \right]$$

$$\Gamma_\mu \equiv \Gamma^\lambda_{\lambda\mu}$$

$$\Rightarrow \partial_\lambda \Gamma^\lambda_{\mu\nu} - \frac{1}{2} \partial_\nu \Gamma_\mu - \frac{1}{2} \partial_\mu \Gamma_\nu = 0$$

$$- \partial_\lambda H^{\mu\nu} + \frac{1}{2} \delta^\nu_\lambda \partial_\rho H^{\rho\lambda} + \frac{1}{2} \delta^\mu_\lambda \partial_\rho H^{\rho\nu} +$$

$$+ \eta^{\mu\nu} \Gamma_\lambda + \frac{1}{2} \eta^{\rho\sigma} \Gamma^\nu_{\rho\sigma} \delta^\mu_\lambda + \frac{1}{2} \eta^{\rho\sigma} \Gamma^\mu_{\rho\sigma} \delta^\nu_\lambda -$$

$$- \eta^{\mu\sigma} \Gamma^\nu_{\lambda\sigma} - \eta^{\nu\sigma} \Gamma^\mu_{\lambda\sigma} = 0$$

$$\Rightarrow \eta^{\mu\nu} \Gamma^\lambda_{\mu\nu} = - \partial_\mu H^{\mu\lambda}$$

$$\Rightarrow - \partial_\lambda H^{\mu\nu} + \eta^{\mu\nu} \Gamma_\lambda - \eta^{\mu\sigma} \Gamma^\nu_{\lambda\sigma} - \eta^{\nu\sigma} \Gamma^\mu_{\lambda\sigma} = 0$$

$$\Rightarrow \Gamma^\lambda_{\mu\nu} = -\frac{1}{2} (\partial_\mu H^\lambda_\nu + \partial_\nu H^\lambda_\mu - \partial^\lambda H_{\mu\nu}) +$$

$$+ \frac{1}{4} (\delta^\lambda_\mu \partial_\nu H + \delta^\lambda_\nu \partial_\mu H - \eta_{\mu\nu} \partial^\lambda H)$$

$$\Rightarrow \underbrace{\frac{1}{2} (\Box H_{\mu\nu} - \partial_\mu \partial_\lambda H^\lambda_\nu - \partial_\nu \partial_\lambda H^\lambda_\mu - \frac{1}{2} g_{\mu\nu} \Box H)}_{R_{\mu\nu}^{lin}} = 0 \quad (10)$$

To compute EMT of $H_{\mu\nu}$ covariantize

$$g^{\mu\nu} \rightarrow g^{\mu\nu} + \delta\gamma^{\mu\nu}$$

N.B. This is just a mathematical tool to build conserved symmetric EMT. We could use instead the Noether EMT and symmetrize it.

Beware : a) covariant derivatives of $\Gamma^\lambda_{\mu\nu} \Rightarrow$ take into account the Christoffels $C^\lambda_{\mu\nu}[\gamma]$

b) $H^{\mu\nu}$ is taken to transform as a contravariant density

$$S^{(2)} = \int d^4x \left[H^{\mu\nu} (\nabla_\lambda \Gamma^\lambda_{\mu\nu} - \nabla_\nu \Gamma^\lambda_\mu) + \right. \\ \left. + \sqrt{-\gamma} \gamma^{\mu\nu} (\Gamma^\lambda_{\mu\nu} \Gamma_\lambda - \Gamma^\lambda_{\mu\rho} \Gamma^\rho_{\nu\lambda}) \right]$$

no $\sqrt{-\gamma}$ here!

Ex 17 Derive:

(11)

$$\bar{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda{}_\lambda =$$

$$= (\Gamma^\lambda{}_\lambda \Gamma^\mu{}_\mu - \Gamma^\mu{}_\lambda \Gamma^\lambda{}_\mu) - \sigma_{\mu\nu}$$

$$\begin{aligned} \bar{T}_{\mu\nu} = \frac{1}{2} \partial_\alpha \Big[& g_{\mu\nu} (H^{\lambda\rho} \Gamma^\rho{}_\lambda - \frac{1}{2} H^\lambda{}_\lambda \Gamma^\rho{}_\rho) + \\ & + (H^{\mu\nu} \Gamma^\alpha{}_\alpha - H^{\mu\alpha} \Gamma^\nu{}_\nu - H^{\nu\alpha} \Gamma^\mu{}_\mu) + \\ & + H^{\alpha\beta} (\Gamma^\mu{}_{\beta\nu} + \Gamma^\nu{}_{\beta\mu}) + H^{\mu\alpha} (\Gamma^\alpha{}_{\rho\nu} - \Gamma^\nu{}_{\alpha\rho}) + \\ & + H^{\nu\rho} (\Gamma^\alpha{}_{\rho\mu} - \Gamma^\mu{}_{\alpha\rho}) \Big] \end{aligned}$$

comes from variation of $\mathcal{L}^\lambda{}_\mu[\varphi]$

Claim: The equations $R_{\mu\nu}^{\text{lin}} = -\bar{T}_{\mu\nu}$ follow from the action (x)

$$S = S^{(2)} + \int d^4x H^{\mu\nu} (\Gamma^\lambda{}_\lambda \Gamma^\mu{}_\mu - \Gamma^\mu{}_\lambda \Gamma^\lambda{}_\mu)$$

$$\begin{aligned} \Rightarrow \partial_\alpha \Gamma^\lambda{}_\lambda \Gamma^\mu{}_\mu - \frac{1}{2} \partial_\nu \Gamma^\mu{}_\mu - \frac{1}{2} \partial_\mu \Gamma^\nu{}_\nu + \Gamma^\lambda{}_\lambda \Gamma^\mu{}_\mu - \Gamma^\mu{}_\lambda \Gamma^\lambda{}_\mu = 0 \\ - \partial_\alpha H^{\mu\nu} + (g^{\mu\nu} + H^{\mu\nu}) \Gamma^\alpha{}_\alpha - (g^{\mu\sigma} + H^{\mu\sigma}) \Gamma^\nu{}_\sigma - \\ - (g^{\nu\sigma} + H^{\nu\sigma}) \Gamma^\mu{}_\sigma = 0 \end{aligned} \quad (**) \quad \left. \vphantom{\begin{aligned} \Rightarrow \partial_\alpha \Gamma^\lambda{}_\lambda \Gamma^\mu{}_\mu - \frac{1}{2} \partial_\nu \Gamma^\mu{}_\mu - \frac{1}{2} \partial_\mu \Gamma^\nu{}_\nu + \Gamma^\lambda{}_\lambda \Gamma^\mu{}_\mu - \Gamma^\mu{}_\lambda \Gamma^\lambda{}_\mu = 0 \\ - \partial_\alpha H^{\mu\nu} + (g^{\mu\nu} + H^{\mu\nu}) \Gamma^\alpha{}_\alpha - (g^{\mu\sigma} + H^{\mu\sigma}) \Gamma^\nu{}_\sigma - \\ - (g^{\nu\sigma} + H^{\nu\sigma}) \Gamma^\mu{}_\sigma = 0 \end{aligned}} \right\}$$

$\Rightarrow \Gamma^\lambda{}_\lambda$ - Christoffel of the total metric
 $g^{\mu\nu} + H^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$

$$\boxed{R_{\mu\nu}^{\text{total}} = 0}$$

Ex 18 Check that Eqs (*) and (**) are equivalent.

(12)

Important: There are no further contributions to $T_{\mu\nu}$:

$$S^{(3)} = \int d^4x H^{\mu\nu} (r_{\mu\lambda}^{\lambda} r_{\lambda}^{\lambda} - r_{\rho\mu}^{\lambda} r_{\lambda\nu}^{\rho})$$

does not contain $\psi^{\mu\nu}$ because $H^{\mu\nu}$ is a density!

\Rightarrow The procedure stops at this order and we recover GR.

8) Matter couplings

$$\text{Eq. of motion: } \frac{\delta S_{GR}}{\delta H^{\mu\nu}} + \underbrace{\frac{\delta S_{mat}}{\delta H^{\mu\nu}}}_{T_{\mu\nu}^{mat}} = 0$$

On the other hand,

$$T_{\mu\nu}^{mat} = \frac{\delta S_{mat} [\varphi, h_{\mu\nu}, r_{\mu\nu}^{\lambda}, y^{\mu\nu} + \delta y^{\mu\nu}]}{\delta y^{\mu\nu}} \bigg|_{\delta y^{\mu\nu} = 0}$$

$$\Rightarrow \frac{\delta S_{mat}}{\delta y^{\mu\nu}} \bigg|_{\delta y = 0} = \frac{\delta S_{mat}}{\delta h^{\mu\nu}}$$

$$\Rightarrow S_{mat} [\varphi, y^{\mu\nu} + h^{\mu\nu}]$$

N.B. This is an independent derivation of the equivalence principle.

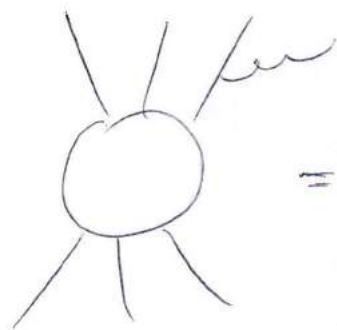
#E. Recent developments

(13)

9) Cachazo, Strominger, 1404. 4091

"Evidence for a New Soft Graviton Theorem"

C. White 1103. 2981, 1406. 7189



$$= S^{(-1)} + S^{(0)} + \dots$$

$$\sum_n \frac{(\tilde{\epsilon}_\mu^* p_\mu^{(n)})^2}{2(p \cdot k)} y_n S_{hard}$$

Conjecture:

$$S^{(0)} \propto \sum_n \frac{\tilde{\epsilon}_\mu^* \tilde{\epsilon}_\nu p_\mu^{(n)} k_\rho j^{(n)} p_\nu}{(p_n \cdot k)} S_{hard}$$

$$p_n^\rho \frac{\partial}{\partial p_n^\nu} - p_n^\nu \frac{\partial}{\partial p_n^\rho} = i \Sigma_{\nu\rho}$$

total angular momentum
of the n -th particle