Lecture 4

II Relation between thermodynamics and gravity.

literature: J. Bekenstein, PRD, 7, 2333 (1973)
PRD, 9, 3293 (1974)

Birrell, Davies "Quantum Fields in Curved Space", Cambridge U. Press, 1982

1) General discussion: Hawking radiation $\Rightarrow$
$\Rightarrow$ thermodynamics $\Rightarrow$ evaporation of black holes
$\Rightarrow$ information paradox

2) Classical gravity: Penrose diagrams

a) flat space-time

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(x)\, g_{\mu\nu}$$

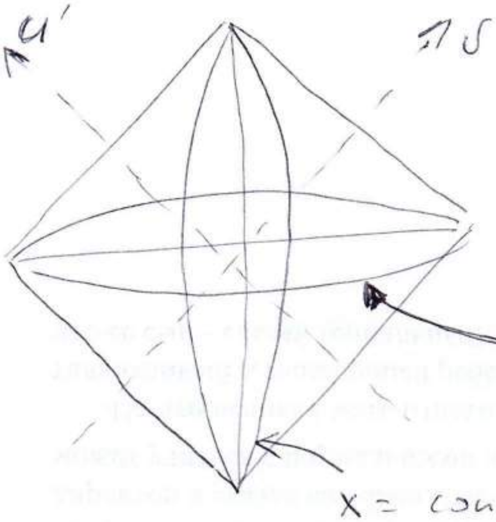
$$ds^2 = -dt^2 + dx^2 = du\, dv$$

$u = t - x \qquad v = t + x$

$$u = tg\,\frac{u'}{2}, \quad v = tg\,\frac{v'}{2}$$

$$-\pi \le u', v' \le \pi$$

$$ds^2 = \underbrace{\frac{1}{4\cos^2\frac{u'}{2}\cos^2\frac{v'}{2}}}_{=\Omega^2}\, du'dv' \xrightarrow[\text{transf.}]{\text{conformal (Weyl)}} du'dv'$$



rays propagate at 45°

<u>More than d=2</u>

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

every point corresponds to a 2-sphere

light rays "reflect" from r=0

6) Schwarzschild solution

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega^2$$

$$G = \hbar = c = 1$$

Kruskal coordinates (regular at $r = 2M$)

$$u = t - r_*\ ;\quad v = t + r_*$$

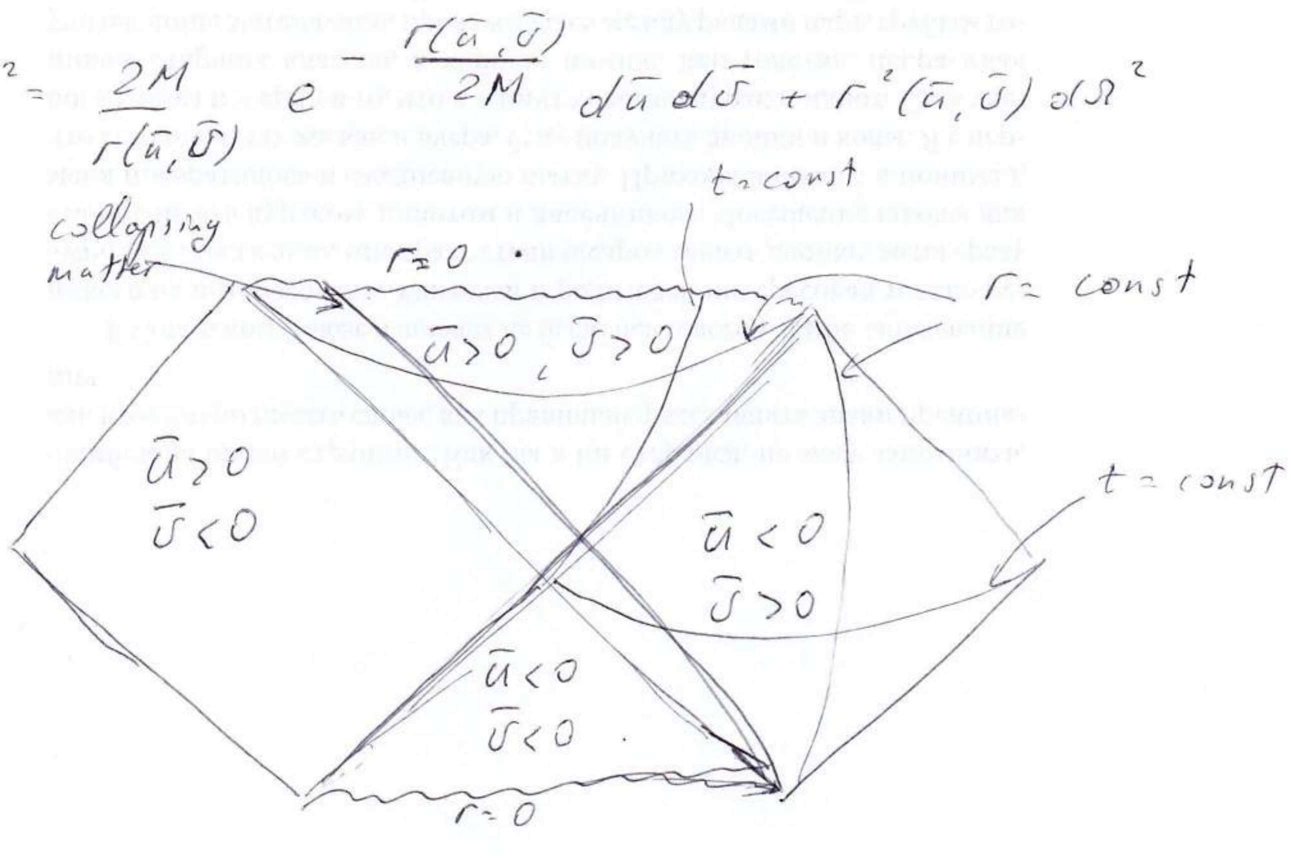
$$r_* = \int \frac{dr}{1-\frac{2M}{r}} = r + 2M\ln\left(\frac{r}{2M} - 1\right)$$

$$\Rightarrow\ ds^2 = \left(1-\frac{2M}{r}\right)du\,dv + r^2 d\Omega^2$$

$$\bar{u} = -4M e^{-\frac{u}{4M}}\ ;\quad \bar{v} = 4M e^{\frac{v}{4M}}$$

$\bar{u} < 0,\ \bar{v} > 0$ – asymptotic region

$$ds^2 = \frac{2M}{r(\bar{u},\bar{v})} e^{\frac{r(\bar{u},\bar{v})}{2M}} d\bar{u}\,d\bar{v} + r^2(\bar{u},\bar{v})\,d\Omega^2$$



Ex. 19 Draw the Penrose diagram for the maximal extension of the inflationary metric (de Sitter):

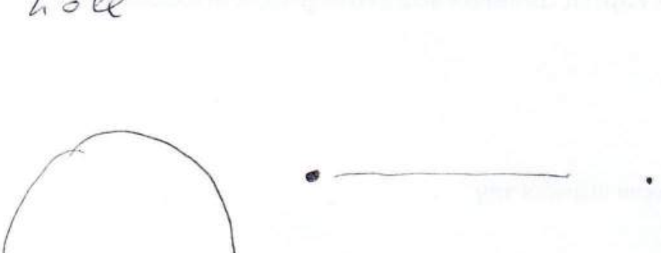
$$ds^2 = \frac{-dy^2 + \sum_{i=1}^{3} dx_i^2}{y^2}$$

3) Black hole entropy

A puzzle: entropy disappears inside black hole $\Rightarrow$ violation of the second law of thermodynamics?

On the other hand, theorems from GR: in all classical processes the area of horizon increases

Take a particle and lower it into a black hole



$$-E^2 g^{tt} + \underbrace{p_r^2 g^{rr}}_{0} = \mu^2$$

$$E = \mu \left(1 - \frac{2M}{r}\right)^{1/2}$$

If r down to $2M \Rightarrow$ no change of the BH mass. But recall the uncertainty principle:

$$\delta l \gtrsim \frac{1}{\mu}$$

$$dl^2 = \frac{dr^2}{1-\frac{2M}{r}} \Rightarrow \delta l^2 \simeq \frac{\delta r^2}{\delta r} \cdot 2M = \delta r \cdot 2M$$

$$E \sim \mu \left(\frac{\delta r}{2M}\right)^{\frac{1}{2}} \simeq \frac{\mu}{2M} \delta l \gtrsim \frac{1}{2M}$$

$$\Rightarrow \Delta M \gtrsim \frac{1}{2M} \Rightarrow \Delta (M^2) \gtrsim 1$$

BH swallowed a bit of information (particle exist or not)

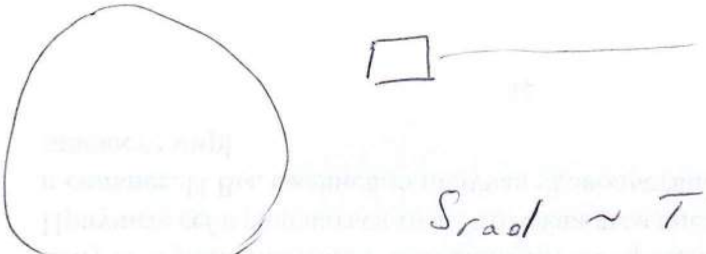
$$\Rightarrow \Delta S \gtrsim \ln 2$$

$\Rightarrow$ natural identification

$S_{BH} \propto M^2$ the exact formula fixed by Hawking: $S_{BH} = \pi M^2$

N.B. No elementary particles with large spin: the size ~~spin~~ must grow as the spin increases (true for strings)

More checks: lowering a macroscopic system (box with radiation)



$$S_{rad} \sim T^3 L^3$$

$$M_{rad} \sim T^4 L^3$$

$$E_{rad} \sim T^4 L^3 \left(\frac{\delta r}{2M}\right)^{1/2} \geq T^4 L^3 \cdot \frac{L}{2M}$$

$$= \frac{S_{rad}}{2M} \underbrace{(T \cdot L)}_{\geq 1} \geq \frac{S_{rad}}{2M}$$

$$\Rightarrow \Delta S_{BH} \geq S_{rad} \quad \underline{\text{O.K.}}$$

4) Quantum fields in curved space: general scheme

$$S = \int d^d x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} (m^2 - \xi R) \varphi^2 \right]$$

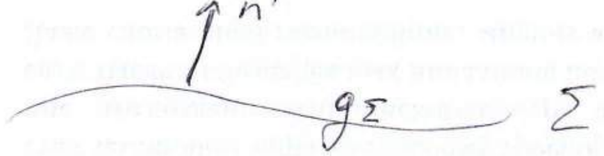
conformal coupling $\xi = \frac{d-2}{4(d-1)}$

equation:

$$(\Box - m^2 + \xi R) \varphi = 0$$

scalar product on solutions

$$\langle \varphi_1, \varphi_2 \rangle = -i \int_\Sigma d\sigma \sqrt{g_\Sigma}\, n^\mu \left(\varphi_1 \partial_\mu \varphi_2^* - \partial_\mu \varphi_1 \cdot \varphi_2^* \right)$$



<u>Ex. 20</u> Show that $\langle \varphi_1, \varphi_2 \rangle$ is independent of the choice of Σ

Complete set of modes : u_i, u_i^*

$$\left.\begin{array}{l} (u_i, u_j) = \delta_{ij} \\ (u_i^*, u_j^*) = -\delta_{ij} \\ (u_i, u_j^*) = 0 \end{array}\right\}$$

analog of plane waves in flat space

$$u_i = \frac{1}{(2\pi)^{3/2}\sqrt{2\omega}}\, e^{-i\omega t + ikx}$$

$$u_i^* = \frac{1}{(2\pi)^{3/2}\sqrt{2\omega}}\, e^{i\omega t - ikx}$$

Expand:

$$\varphi(x) = \sum_i \left(u_i(x)\, a_i + u_i^*(x)\, a_i^+ \right)$$

$$[a_i, a_j^+] = \delta_{ij} \quad ; \quad a_i |0\rangle = 0$$

But the choice of u_i is not uniqe.
Let's take $\tilde{a}_j$:

$$\phi(x) = \sum_j \left(\tilde{u}_j \tilde{a}_j + \tilde{u}_j^* \tilde{a}_j^+ \right) \; , \quad \tilde{a}_j |\tilde{0}\rangle = 0$$

$$\begin{cases} \tilde{u}_j = \sum_i \left(\alpha_{ji} u_i + \beta_{ji} u_i^* \right) \\ \tilde{u}_j^* = \sum_i \left(\alpha_{ji}^* u_i^* + \beta_{ji}^* u_i \right) \Rightarrow \langle \tilde{u}_j^* \, u_i \rangle = \beta_{ji}^* \end{cases}$$

$$\langle \tilde{u}_j , u_i \rangle = \alpha_{ji}$$

$$\langle \tilde{u}_j , u_i^* \rangle = -\beta_{ji}$$

$$\begin{cases} u_i = \sum_j \left(\alpha_{ji}^* \tilde{u}_j - \beta_{ji} \tilde{u}_j^* \right) \\ u_i^* = \sum_j \left(\alpha_{ji} \tilde{u}_j^* - \beta_{ji}^* \tilde{u}_j \right) \end{cases}$$

Completenes relations

$$\sum_j \left(\alpha_{ji}^* \alpha_{jk} - \beta_{ji} \beta_{jk}^* \right) = \delta_{ik}$$

$$\sum_j \left(\alpha_{ji}^* \beta_{jk} - \beta_{ji} \alpha_{jk}^* \right) = 0$$

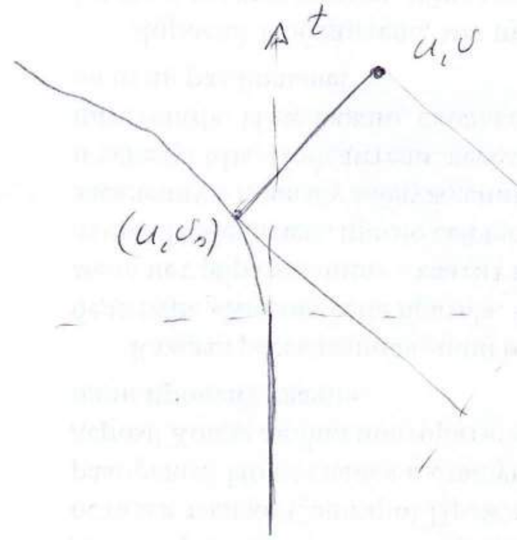
$$\sum_i \left(\alpha_{ji} \alpha_{ki}^* - \beta_{ji} \beta_{ki}^* \right) = \delta_{jk}$$

$$\sum_i \left(\alpha_{ji} \beta_{ki} - \beta_{ji} \alpha_{ki} \right) = 0$$

$$\left.\begin{cases} a_i = \sum_j \alpha_{ji}\, \tilde{a}_j + \beta^*_{ji}\, \tilde{a}^+_j \\ \tilde{a}_j = \sum_i \left(\alpha^*_{ji}\, a_i - \beta^*_{ji}\, a^+_i\right) \end{cases}\right\}$$ Bogolyubov transformation

$$\langle 0 | \tilde{a}^+_j \tilde{a}_j | 0 \rangle = \sum |\beta_{ji}|^2$$ ← vacuum $|0\rangle$ is not empty for $\tilde{a}_j$ modes

5) Moving mirrors in 2 dim.



$x = z(t)$,

$z(t) = 0$ at $t < 0$

$$\frac{\partial^2 \varphi}{\partial u \partial v} = 0 \qquad u = t - x,\quad v = t + x$$

$\varphi(t, z(t)) = 0$ – Dirichlet b.c.

$$u^{in}_\omega = \frac{1}{\sqrt{4\pi\omega}}\left(e^{-i\omega v} - e^{-i\omega(2\tau_u - u)}\right)$$

$u = \tau_u - z(\tau_u)$

$v_0 = \tau_u + z(\tau_u) \Rightarrow v_0 = 2\tau_u - u$

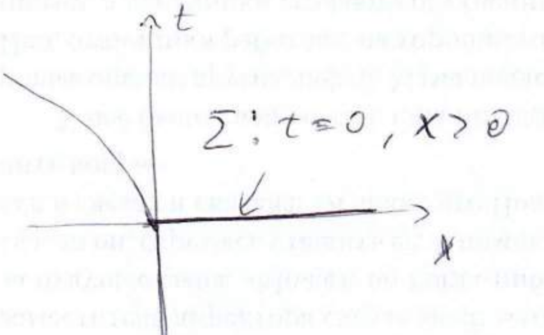
$$u^{out}_\omega = \frac{1}{\sqrt{4\pi\omega}}\left(e^{-i\omega u} - e^{-i\omega(2\tau_v - v)}\right)$$

$\tau_v + z(\tau_v) = v$

$\tau_v - z(\tau_v) = u_0 \Rightarrow u_0 = 2\tau_v - v$

$$\beta_{\omega'\omega} = - \langle u^{out}_{\omega'}, u^{in}_{\omega} \rangle =$$

$$= \int_0^\infty dx \, \frac{1}{2\pi} \sqrt{\frac{\omega}{\omega'}} \left(e^{i(\omega+\omega')x} + e^{+i(\omega'-\omega)x - i2\tau_x \omega'} \right)$$



Examples

a) constant velocity $\Rightarrow z(t) = -\alpha t$

$$\tau_v - \alpha \tau_v = v \Rightarrow \tau_v = \frac{v}{1-\alpha}$$

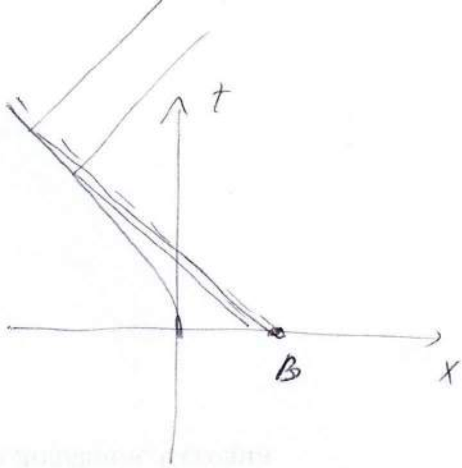
$$\beta_{\omega'\omega} = \frac{\sqrt{\omega\omega'}}{2\pi i} \left(-\frac{2\alpha}{1-\alpha} \right) \frac{1}{(\omega+\omega')\left(\frac{1+\alpha}{1-\alpha}\omega' + \omega\right)}$$

non-zero because of the abrupt change in velocity from 0 to $-\alpha$

$$\int d\omega \, |\beta_{\omega'\omega}|^2 = \frac{1}{\pi^2 \omega'} \left(\frac{1}{2} + \frac{1}{4\alpha} \ln \frac{1+\alpha}{1-\alpha} \right)$$

flat spectrum

b) constant acceleration



$$z = B - (B^2 + t^2)^{1/2}, \quad t > 0$$

$$\tau_x = \frac{2Bx - x^2}{2(B-x)}, \quad x < B$$

$$\beta_{\omega'\omega} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\left(\int_0^{\infty} e^{i(\omega+\omega')x}\,dx + \int_0^B e^{-i\frac{Bx}{B-x}\omega' - i\omega x}\,dx\right)$$

$\Rightarrow$ late-time asymptotics comes from the singularity at $x = B$

$$\beta_{\omega'\omega} \approx \frac{B}{\pi} e^{i(\omega'-\omega)B} K_1\left(2B\sqrt{\omega\omega'}\right)$$

N.B. $K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{dt}{t^{\nu+1}} e^{-t-\frac{z^2}{4t}}$

c) $z \sim B - t - A e^{-2\kappa t}, \quad t \to \infty$

$$\tau_s + B - \tau_s A e^{-2\kappa\tau_s} = s \Rightarrow \tau_s \approx -\frac{1}{2\kappa}\ln\frac{B-s}{A}$$

$$\beta_{\omega'\omega} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\left(\int_0^{\infty} e^{i(\omega+\omega')x}\,dx + \int_0^B e^{i(\omega'-\omega)x + \frac{i\omega'}{\kappa}\ln\frac{B-x}{A}}\,dx\right)$$

$$\approx \frac{1}{2\pi i}\sqrt{\frac{\omega}{\omega'}}\,\frac{e^{-\frac{\pi\omega'}{2\kappa}}}{\omega-\omega'}\left(A(\omega-\omega')\right)^{-\frac{i\omega'}{\kappa}} e^{i(\omega'-\omega)B} \times$$

$$\times\, \Gamma\left(1 + \frac{i\omega'}{\kappa}\right)$$

$$|\beta_{\omega'\omega}|^2 = \frac{1}{(2\pi)^2} \frac{\omega e^{-\frac{\pi\omega'}{\varkappa}}}{\omega'(\omega-\omega')^2} \Gamma\left(1+\frac{i\omega'}{\varkappa}\right)\Gamma\left(1-\frac{i\omega'}{\varkappa}\right)$$

use $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$

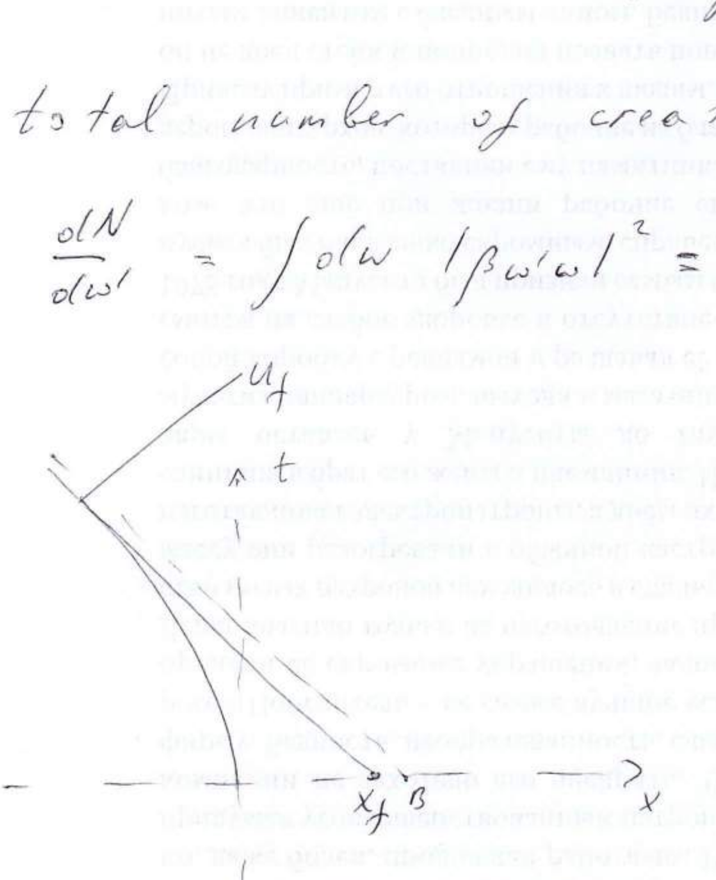
$$\Rightarrow |\beta_{\omega\omega'}|^2 = \frac{1}{4\pi\varkappa}\frac{\omega}{(\omega-\omega')^2}\frac{1}{e^{\frac{2\pi\omega'}{\varkappa}}-1}$$

temperature $\boxed{T = \frac{\varkappa}{2\pi}}$

total number of created particles:

$$\frac{dN}{d\omega'} = \int d\omega\, |\beta_{\omega'\omega}|^2 = \frac{1}{4\pi\varkappa}\frac{1}{e^{\frac{2\pi\omega'}{\varkappa}}-1}\int \frac{d\omega}{\omega}$$

diverges because of constant late-time flux



$$u_f = t_f - (B - t_f - Ae^{-2\varkappa t_f}) \simeq 2t_f$$

$$x_f = v_f = t_f + (B - t_f - Ae^{-2\varkappa t_f}) = B - Ae^{-2\varkappa t_f} = B - Ae^{-\varkappa u_f}$$

$B - x_f \sim Ae^{-\varkappa u_f}$ – exponentially small

$\omega(B-x_f) \lesssim 1 \Rightarrow \omega \lesssim \frac{1}{A}e^{\varkappa u_f}$ – exponentially large!

$$\int d\omega\, |\beta_{\omega'\omega}|^2 = \frac{u_f}{4\pi}\frac{1}{e^{2\pi\omega'/\varkappa}-1} \Rightarrow \text{flux} = \frac{d\omega'}{4\pi}\frac{1}{e^{2\pi\omega'/\varkappa}-1}$$