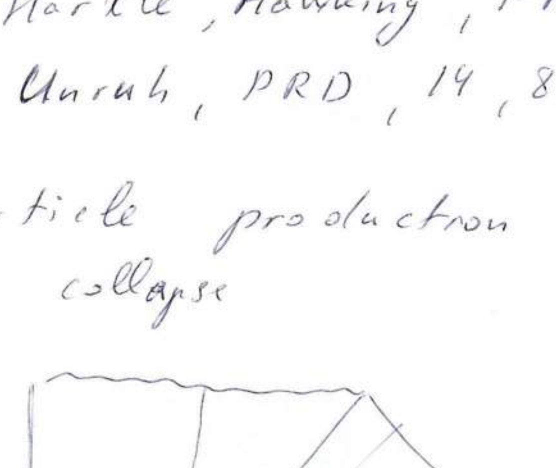
Lecture 5

Refs: Birrell, Davies
Hawking, Commun. Math. Phys 43, 199 (1975)
Hartle, Hawking, PRD, 13, 2188 (1976)
Unruh, PRD, 14, 870 (1976)

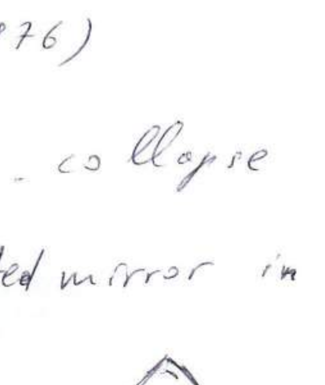
1) Particle production in grav. collapse

collapse

accelerated mirror in Mink.



$\Longleftrightarrow$



$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

last ray

for Schwarzschild
$$f(r) = 1 - \frac{2M}{r}$$

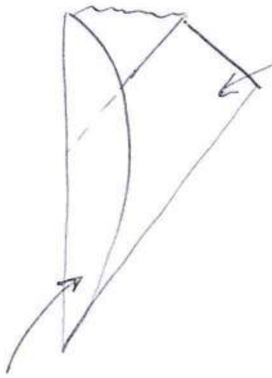
$$\left.\begin{aligned} ds^2 &= -d\tau^2 + \left(\frac{\partial r}{\partial \rho}\right)^2 d\rho^2 + r^2(\tau,\rho)\,d\Omega^2 \\ r(\tau,\rho) &= \left(\frac{9M(\rho)}{2}\left(\tau_0(r)-\tau\right)^2\right)^{1/3} \end{aligned}\right\}\ \text{dust ball}$$

$ds^2 = -d\tau^2 + dr^2 + r^2 d\Omega$ if a thin shell

Internal metric is not important

We take the thin shell for simplicity

2) 2d model



$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} =$$

$$= -f(r)\,du\,dv$$

$$u, v = t \mp r_*$$

$$r_* = \int \frac{dr}{f(r)} = r + 2M\ln\left(\frac{r}{2M} - 1\right)$$

for Schwarzschild

$$ds^2 = -d\tau^2 + dr^2 =$$

$$= -dU\,dV$$

$$U, V = \tau \mp r$$

On the shell: $dU = d\tau - dr = d\tau(1 - \dot r)$

$$dV = d\tau(1 + \dot r)$$

$$du = dt - \frac{dr}{f(r)} = d\tau\left(\frac{dt}{d\tau} - \frac{\dot r}{f(r)}\right)$$

$$-dt^2 f(r) + \frac{dr^2}{f(r)} = -d\tau^2 + dr^2$$

$$\Rightarrow \left(\frac{dt}{d\tau}\right)^2 = \frac{\dot r^2}{f^2} + \frac{1 - \dot r^2}{f} \Rightarrow \frac{dt}{d\tau} = \frac{\sqrt{(1-\dot r^2)f + \dot r^2}}{f}$$

$$\Rightarrow du = d\tau \cdot \frac{\sqrt{\dot r^2 + (1-\dot r^2)f} - \dot r}{f}$$

$$\frac{du}{dU} = \frac{\sqrt{\dot r^2 + (1-\dot r^2)f} - \dot r}{f(r)(1-\dot r)}$$

$$\frac{dv}{dV} = \frac{\sqrt{\dot r^2 + (1-\dot r^2)f} + \dot r}{f(r)(1+\dot r)}.$$

In the vicinity of $r \approx r_h$:

$$r = r_h + A\,(\tau - \tau_h) + \dots \quad , \quad -1 < A < 0$$

$$\frac{dv}{dV} = \frac{-A\left(1 + \frac{1-A^2}{2A^2} f\right) + A}{f\,(1+A)} = -\frac{1-A}{2A} \quad \text{- regular}$$

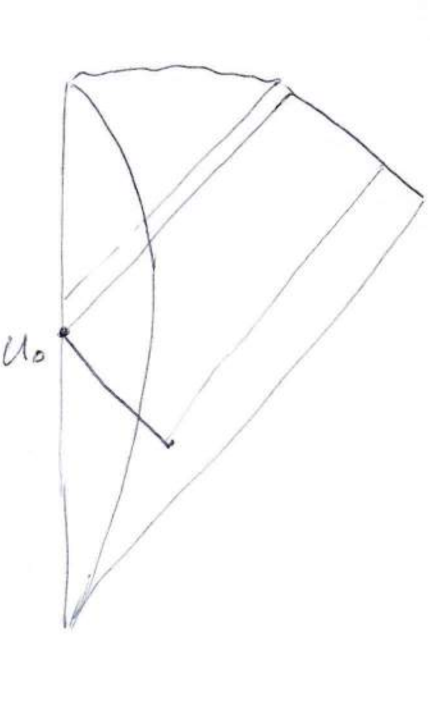
$\Rightarrow$ $v(V)$ is regular at the horizon crossing

$$\frac{du}{dU} = -\frac{2A}{f(r)\,(1-A)} = -\frac{2A}{1-A}\,\frac{1}{f'_h \cdot A\,(\tau - \tau_h)}$$

$$U = \tau - r = \tau - r_h - A\,(\tau - \tau_h) = U_h - A\,(\tau - \tau_h)$$

$$\Rightarrow \frac{du}{dU} = -\frac{2}{f'_h \cdot (U - U_h)} \Rightarrow \boxed{u = u_h - \frac{2}{f'_h} \ln |U - U_h|}$$

N.B. All dependence on A dropped out



$$\varphi_{out} = \frac{1}{\sqrt{4\pi\omega}}\left(e^{-i\omega u} - e^{-i\omega u_0}\right)$$

$$u_0 = -\frac{2}{f'_h} \ln |U_0 - U_h| + \text{const}$$

$$U_0 = V = v + \text{const}$$

$$\Rightarrow u_0 = -\frac{2}{f'_h} \ln |v - v_h| + \text{const}$$

$$\varphi_\omega^{out} = \frac{1}{\sqrt{4\pi\omega}} \left(e^{-i\omega u} \cancel{=} e^{\frac{i 2\omega}{f_h'} \ln|v - v_h| + \text{regular}} \right)$$

The same expression as for the mirror with exponential trajectory from Lecture 4.

$$\Rightarrow \left|\beta_{\omega\omega_{in}}\right|^2 = \frac{1}{\omega_{in}} \frac{1}{\pi f_h'} \frac{1}{e^{\omega/T} - 1}$$

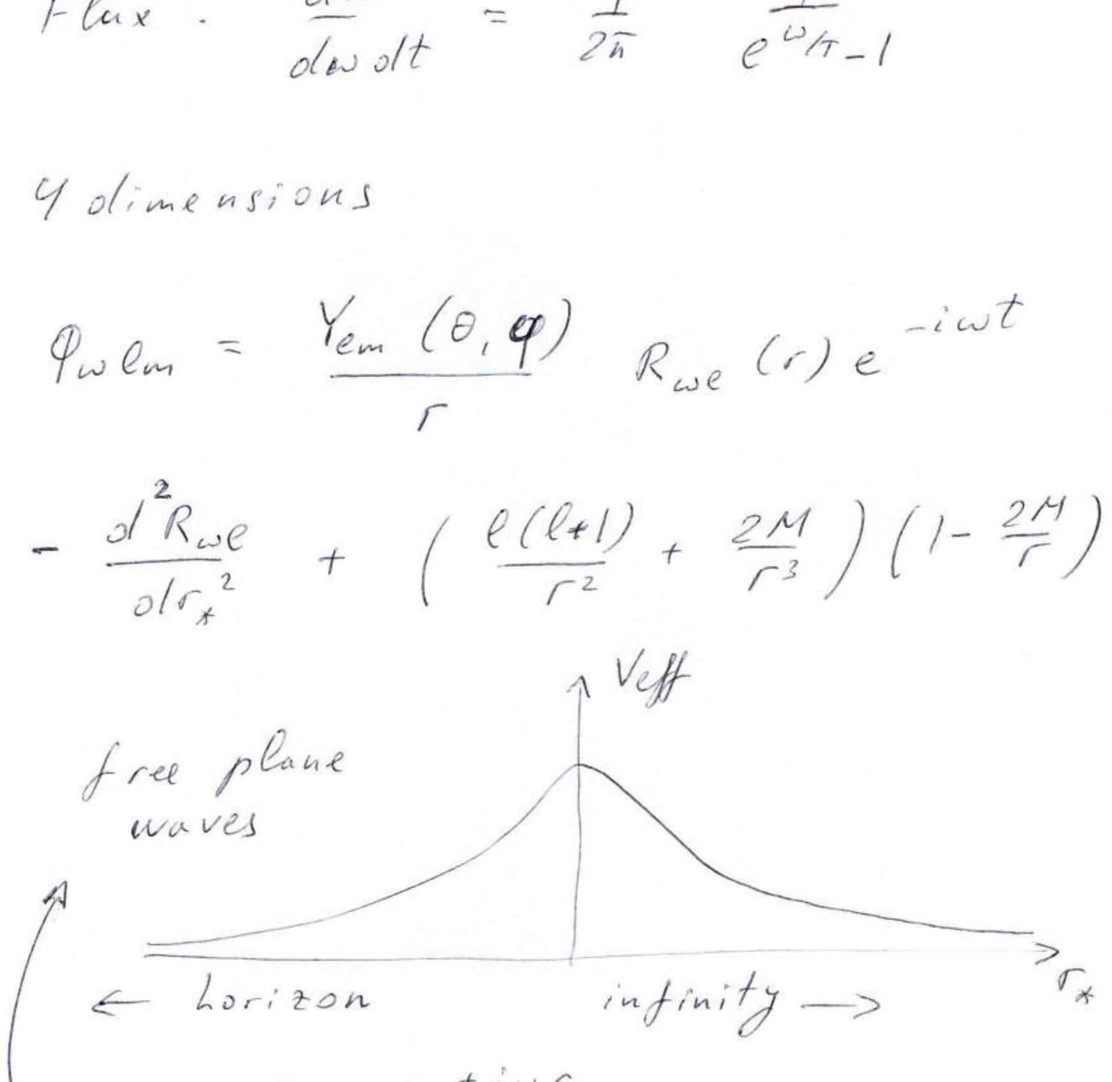
$$\boxed{T = \frac{f_h'}{4\pi} = \frac{1}{8\pi M}}$$

Flux: $\dfrac{dN}{d\omega\, dt} = \dfrac{1}{2\pi} \dfrac{1}{e^{\omega/T} - 1}$

3) 4 dimensions

$$\varphi_{\omega lm} = \frac{Y_{em}(\theta, \varphi)}{r} R_{\omega e}(r) e^{-i\omega t}$$

$$-\frac{d^2 R_{\omega e}}{d r_*^2} + \left(\frac{e(e+1)}{r^2} + \frac{2M}{r^3} \right)\left(1 - \frac{2M}{r}\right) R_{\omega e} = \omega^2 R_{\omega e}$$



$$R_{\omega e} = \frac{1}{\sqrt{4\pi\omega}} e^{\pm i\omega r_*}$$

$$\frac{dN_{em}}{d\omega\, dt} = \frac{\Gamma_{\omega l}}{2\pi} \frac{1}{e^{\omega/T}-1}$$

transmission coefficient
(a.k.a. grey-body factor)

The spectrum is <u>not</u> Planckian. But the BH is in equilibrium with thermal bath

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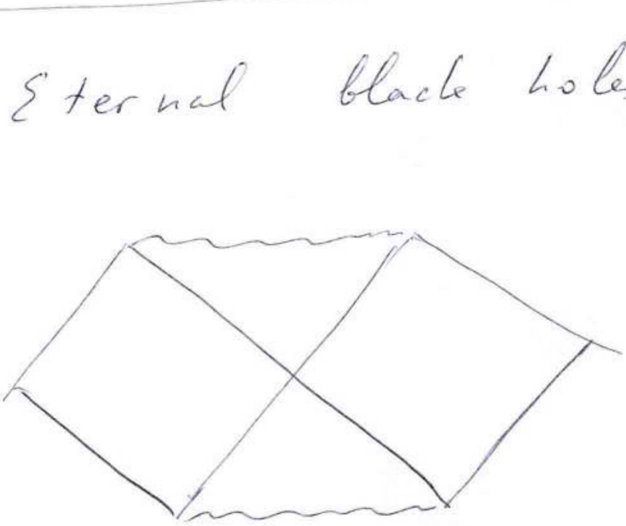
4) Eternal black holes



$$ds^2 = -\left(1-\frac{2M}{r}\right) du\, dv$$
$$= -\frac{2M}{r} e^{-\frac{r}{2M}} d\bar{u}\, d\bar{v}$$
$$\bar{u} = -4M e^{-u/4M}$$
$$\bar{v} = 4M e^{v/4M}$$

choice of vacuum = choice of modes

$$\varphi^B_{\omega,R} = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega u} \; ; \; \varphi^B_{\omega,L} = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v}$$

plane waves at $r \to \infty$
natural vacuum at $r \to \infty$
singular at $r = 2M$

$a^B |0\rangle_B = 0$ – Boulware vacuum

$$\varphi^{H}_{\omega,R} = \frac{1}{\sqrt{4\pi\omega}}\, e^{-i\omega \bar{u}} \;;\; \varphi^{H}_{\omega,L} = \frac{1}{\sqrt{4\pi\omega}}\, e^{-i\omega \bar{v}}$$

- regular at the horizon

$a^{H}_{\omega}\,|0\rangle_{H} = 0$ — Hartle - Hawking vacuum

It is not empty:

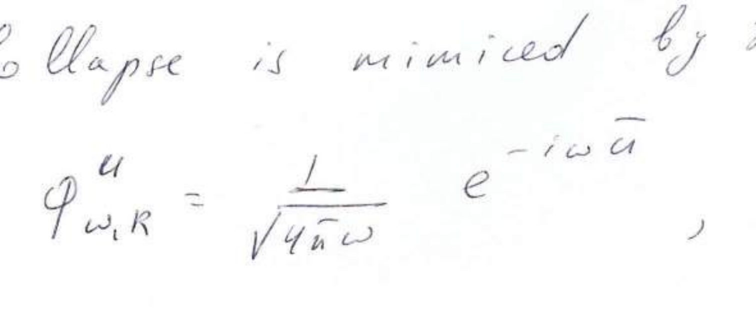
$$\beta_{\omega\omega',RR} = \beta^{*}_{\omega\omega',LL} = \frac{1}{2\pi}\sqrt{\frac{\omega}{\omega'}}\left(\frac{-i}{\omega}\right)\left(\frac{i}{4\pi\omega}\right)^{i\omega' 4M}\Gamma(1+i4M\omega)$$

Thermal fluxes from and <u>to</u> black hole

= BH in thermal bath

Collapse is mimiced by the Unruh vacuum:

$$\varphi^{U}_{\omega,R} = \frac{1}{\sqrt{4\pi\omega}}\, e^{-i\omega \bar{u}} \;,\; \varphi^{U}_{\omega,L} = \frac{1}{\sqrt{4\pi\omega}}\, e^{-i\omega v}$$



5) Detector response

$$x^\mu(\tau) \longrightarrow S_{int} = \lambda \int d\tau \, \varphi(x(\tau)) \, m(\tau)$$

operator acting on the detector degrees of freedom

transition amplitude

$$\mathcal{A} = i\lambda \langle E, \psi | \int_{-\infty}^{\infty} d\tau \, m(\tau) \, \varphi(x(\tau)) | 0, E_0 \rangle$$

ground state

$$m(\tau) = e^{iH_0 \tau} m(0) e^{-iH_0 \tau}$$

$$\Rightarrow \mathcal{A} = i\lambda \langle E | m(0) | E_0 \rangle \langle \psi | \int_{-\infty}^{\infty} d\tau \, e^{i(E-E_0)\tau} \varphi(x(\tau)) | 0 \rangle$$

If we sum over all quanta ψ:

$$P_{EE_0} = \sum_\psi |\mathcal{A}_{E\psi, E_0}|^2 = \lambda^2 |\langle E | m(0) | E_0 \rangle|^2 \cdot$$

$$\underbrace{\cdot \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \, e^{-i\Delta E(\tau_1 - \tau_2)} \underbrace{\langle 0 | \varphi(x(\tau_1)) \varphi(x(\tau_2)) | 0 \rangle}_{G^+(x(\tau_1), x(\tau_2))}}_{\text{detector response function}}$$

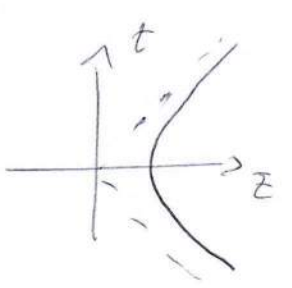
If $G^+(x(\tau_1), x(\tau_2)) = g(\tau_1 - \tau_2)$

$\Rightarrow$ transition rate per unit proper time:

$$\frac{dP_{EE_0}}{d\tau} = \lambda^2 \left| \langle E | m(0) | E_0 \rangle \right|^2 \int_{-\infty}^{\infty} d\tau \, e^{-i(E-E_0)\tau} g(\tau)$$

Example: uniformly accelerated detector

$$\left. \begin{array}{l} x = y = 0 \\ z = \sqrt{t^2 + B^2} \end{array} \right\} \Rightarrow \begin{array}{l} t = B \, \mathrm{sh} \frac{\tau}{B} \\ z = B \, \mathrm{ch} \frac{\tau}{B} \end{array}$$

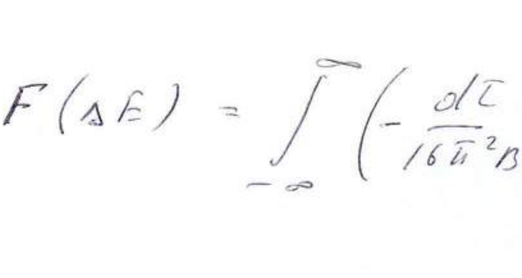


$$G^+(x, x') = - \frac{1}{4\pi^2 \left((t - t' - i\varepsilon)^2 - |\vec{x} - \vec{x}'|^2 \right)}$$

$$G^+(x(\tau), x'(\tau')) = - \frac{1}{16\pi^2 B^2} \frac{1}{\mathrm{sh}^2 \left(\frac{\tau - \tau'}{2B} - i\varepsilon \right)}$$

$$F(\Delta E) = \int_{-\infty}^{\infty} \left(- \frac{d\tau}{16\pi^2 B^2} \right) \frac{e^{-i\Delta E \tau}}{\mathrm{sh}^2 \left(\frac{\tau}{2B} - i\varepsilon \right)}$$

$\Delta E > 0$



$$F(\Delta E) = -\frac{1}{16\pi^2 B^2} \sum_{n=1}^{\infty} (-2\pi i) \left((z + i2\pi B n)^2 \frac{e^{-i\Delta E z}}{sh^2\left(\frac{z}{2B}\right)} \right)' \Bigg|_{z=-i2\pi B n}$$

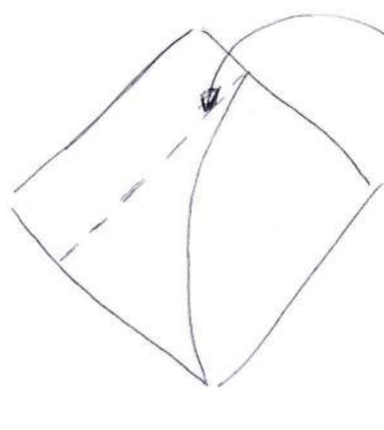
$$= \frac{\Delta E}{2\pi} \sum_{n=1}^{\infty} e^{-2\pi B \Delta E n} = \frac{\Delta E}{2\pi} \underbrace{\frac{1}{e^{2\pi B \Delta E} - 1}}_{T = \frac{1}{2\pi B}}$$

Effect Unruh

Ex. 21 Compare with the response function of an inertial detector in thermal bath

N.B. Excitation of the detector is accompanied by emission of a φ-particle

Ex. 22 Find distribution of momenta of emitted particles accompanying excitations of the detector



causal horizon of the accelerated observer

6) Detector response in Hartle-Hawking vacuum:

$$G^+(t,r;t',r') = -\frac{1}{4\pi} \ln\left((\bar{u}-\bar{u}')(\bar{v}-\bar{v}')\right)$$

$r' \doteq r$

$$\bar{u}-\bar{u}' = -4M\left(e^{-\frac{u}{4M}} - e^{-\frac{u'}{4M}}\right) = -4M e^{\frac{r_*}{4M}}\left(e^{-\frac{t}{4M}} - e^{-\frac{t'}{4M}}\right)$$

$$\bar{v}-\bar{v}' = 4M e^{\frac{r_*}{4M}}\left(e^{\frac{t}{4M}} - e^{\frac{t'}{4M}}\right)$$

$$\Rightarrow G^+(t,r;t',r) = -\frac{1}{4\pi} \ln\left(1-\operatorname{ch}\frac{t-t'}{4M}\right) + \tilde{G}(r) =$$

$$= -\frac{1}{4\pi} \ln\left(1-\operatorname{ch}\frac{\tau-\tau'}{4M\sqrt{1-\frac{2M}{r}}}\right) + \tilde{G}(r)$$

periodic in τ with $\Delta\tau = i\, 8\pi M \sqrt{1-\frac{2M}{r}}$

$$\Rightarrow \text{temperature} \quad T(r) = \frac{1}{8\pi M \underbrace{\sqrt{1-\frac{2M}{r}}}_{\text{blue-shift}}}$$

On the other hand, at fixed $\bar{r}$ ($\approx$ free falling observer)

$$G^+(\bar{t},\bar{r};\bar{t}',\bar{r}) = \underbrace{-\frac{1}{2\pi} \ln(\bar{t}-\bar{t}')}_{}$$

the same as in flat space

$\Rightarrow$ no detection of particles

Ex. 23 Consider de Sitter space-time with the metric

$$ds^2 = \frac{-dy^2 + dx_i^2}{H^2 y^2} \quad , \quad -\infty < y < 0$$

Choose the Bunch - Davies vacuum defined by the modes

φ_ω^{BD} with the asymptotics

$\varphi_\omega^{BD} \propto e^{-i\omega y}$ at $y \to -\infty$

Find the response function of a comoving detector and interprete the result in terms of the causal structure of the space-time.