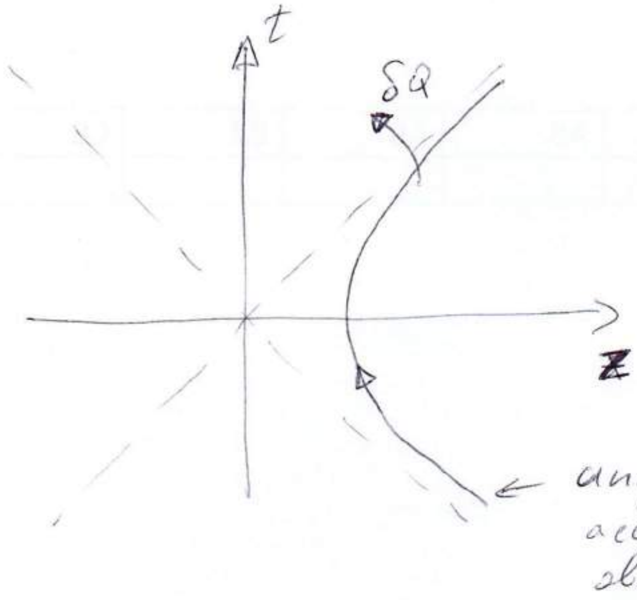
Lecture 7

Ref. Jacobson, PRL, 75, 1260 (1995)

1) From thermodynamics to gravity



$t = B\, sh \frac{\tau}{B}$

$z = B\, ch \frac{\tau}{B}$

B^{-1} - acceleration

$T = \frac{1}{2\pi B}$, $S = \frac{A}{4 l_p^2}$

From GR;

$\delta Q = T dS$ if the process is adiabatic

2) Let us turn the argument around and see what we get

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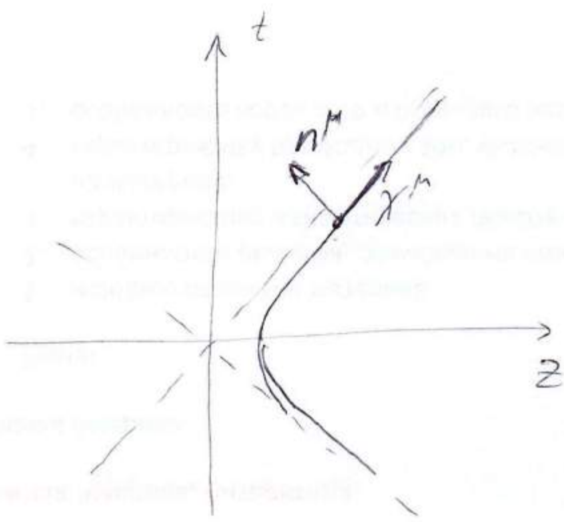


Choose local Lorentz frame ($g_{\mu\nu} = \eta_{\mu\nu}$, $\Gamma_{\mu\nu}^{\lambda} = 0$) and consider a horizon "in equilibrium":

$$\partial_\mu k_\nu = \nabla_\mu k_\nu = 0$$

light-like tangent to the horizon

χ_μ - approximate local Killing vector

$$\nabla_\mu \chi_\nu + \nabla_\nu \chi_\mu = 0 \quad , \quad x_\mu \to x_\mu + \chi_\mu \cdot \varepsilon \text{ - isometry}$$

$$\chi^t = \underbrace{\frac{z}{\sqrt{z^2-t^2}}}_{ch\,\tau/B}, \quad \chi^z = \underbrace{\frac{t}{\sqrt{z^2-t^2}}}_{sh\,\tau/B} \Rightarrow \chi^\mu \underset{B\to 0}{\to} \frac{1}{2} e^{\tau/B} \underbrace{(1,1)}_{k^\mu} = \frac{1}{2} e^{\tau/B} k^\mu$$

$n^t = -sh\frac{\tau}{B}$, $n^z = -ch\frac{\tau}{B}$ - normal vector to the trajectory

$$n^\mu \underset{B\to 0}{\to} -\frac{1}{2} e^{\tau/B}(1,1) = -\frac{1}{2} e^{\tau/B} k^\mu$$

$$\delta Q = \int d\tau\, dA \cdot n^\mu T_{\mu\nu} \chi^\nu \xrightarrow{B \to 0} -\int dA\, d\tau\, \frac{1}{4} e^{\frac{2\tau}{B}} k^\mu k^\nu T_{\mu\nu}$$

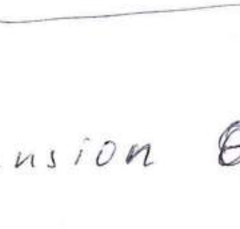
trasverse area   matter EMT

$$t = \frac{B}{2} e^{\tau/B} \Rightarrow \delta Q \simeq -\frac{1}{B}\int dA\, dt \cdot t\, k^\mu k^\nu T_{\mu\nu} \Leftrightarrow$$

$$\Rightarrow \frac{\delta Q}{T} = -2\pi \int dA\, t\, dt\, k^\mu k^\nu T_{\mu\nu}$$

affine parameter along the horizon

3) Expansion θ



$$l^{\mu'} = l^\mu + \left(k^\mu(x_2) - k^\mu(x_1)\right) d\lambda =$$

$$= l^\mu + l^\nu \nabla_\nu k^\mu\, d\lambda$$

$$\sqrt{l^{\mu' 2}} - \sqrt{l^{\mu 2}} = \frac{l^\mu l^\nu \nabla_\nu k_\mu\, d\lambda}{|l|}$$

$$\theta = \nabla_\mu k^\mu = g^\mu_\nu \nabla_\mu k^\nu = \left(k^\mu q_\nu + k_\nu q^\mu + \sum_{i=1}^{2} \frac{l_i^\mu l_{i\nu}}{|l_i|^2}\right)\nabla_\mu k^\nu$$

$$q_\nu^2 = 0$$

$$k_\nu q^\nu = -2$$

$$q^\mu l_{i\mu} = 0$$

$$k^\mu l_{i\mu} = 0$$

$k^\mu \nabla_\mu k^\nu = 0$ – geodesic

$k_\nu \nabla_\mu k^\nu = 0$ because $k_\nu k^\nu = 0$

$$\Rightarrow \theta = \sum_{i=1}^{2} \frac{l_i^\mu l_{i\nu}}{|l_i|^2} \nabla_\mu k^\nu$$

$$\delta A = \delta(|l_1||l_2|) = \sum_i \frac{l_i^\mu l_i^\nu}{|l_i|^2} \nabla_\nu k_\mu \cdot d\lambda \cdot |l_1||l_2| =$$

$$= A \, d\lambda \, \theta \rightarrow \int \theta \, d\lambda \, dA$$

$$\theta = \lambda \underbrace{k^\mu \partial_\mu \theta}$$

$$\hookrightarrow k^\mu \nabla_\mu \nabla_\nu k^\nu = k^\mu \nabla_\nu \nabla_\mu k^\nu - k^\mu R_{\nu\mu} k^\nu =$$

$$= \nabla_\nu (\underbrace{k^\mu \nabla_\mu k^\nu}_{0}) - \underbrace{\nabla_\nu k^\mu \nabla_\mu k^\nu}_{0} - R_{\mu\nu} k^\mu k^\nu$$

$$\Rightarrow \underline{\delta A = - \int \lambda \, d\lambda \, dA \, k^\mu k^\nu R_{\mu\nu}}$$

4) Postulate : $\frac{\delta Q}{T} = \delta S$, $S = \frac{A}{4 l_p^2}$

$$\int \lambda \, d\lambda \, dA \, k^\mu k^\nu (-2\pi T_{\mu\nu}) = \int \lambda \, d\lambda \, dA \, k^\mu k^\nu \left(- \frac{R_{\mu\nu}}{4 l_p^2}\right)$$

$$\int \lambda \, d\lambda \, dA \, k^\mu k^\nu \left(- \frac{R_{\mu\nu}}{4 l_p^2} + 2\pi T_{\mu\nu} \right) = 0$$

This holds for any light-like vector k^μ

$$\Rightarrow R_{\mu\nu} + f g_{\mu\nu} = 8\pi l_p^2 T_{\mu\nu}$$

$$\underbrace{\nabla_\mu R_{\mu\nu}}_{\frac{1}{2}\nabla_\nu R} + \nabla_\nu f = 8\pi l_p^2 \nabla_\nu T_{\mu\nu} = 0$$ due to EMT conservation

$$\Rightarrow f = -\frac{1}{2} R + \underset{\substack{\uparrow \\ \text{constant}}}{\Lambda}$$

$$\Rightarrow \underline{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi l_p^2 T_{\mu\nu}}$$

Conclusion: From postulating thermodynamics of horizons we derived the Einstein's equations