Lecture 8

III Theory of primordial cosmological perturbations

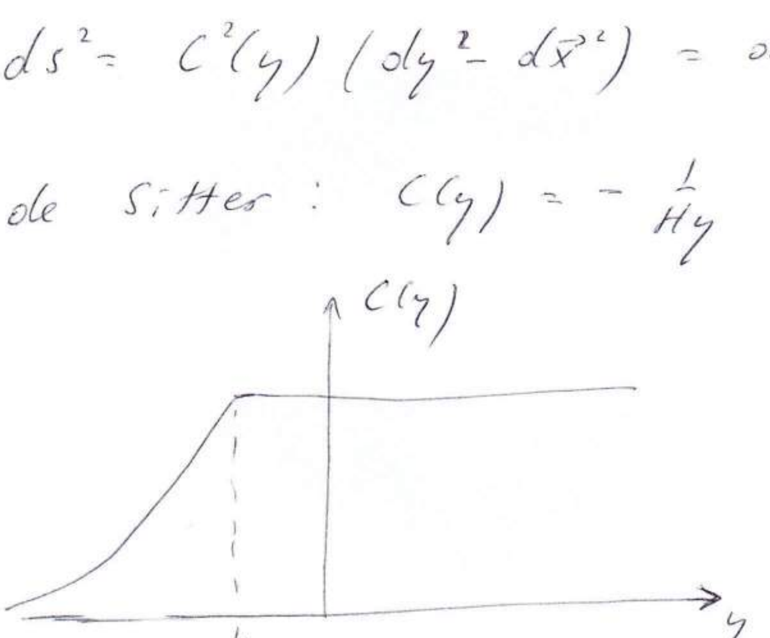
Refs: Polarski, Starobinsky, Class. Quant. Grav, 13, 377 (1996)

1) Toy model for reheating

$$S = -\int d^4x \sqrt{-g}\, (\partial_\mu \varphi)^2$$

$$ds^2 = C^2(\eta)\,(d\eta^2 - d\vec{x}^2) = dt^2 - C^2(t)\,dx^2$$

de Sitter : $C(\eta) = -\frac{1}{H\eta}$



Instantaneous reheating. OK because the field is constant outside horizon

$$\varphi = \frac{1}{C(\eta)} \int d^3k \left(f_k(\eta)\, e^{ikx} a_k + f_k^*(\eta)\, e^{-ikx} a_k^+ \right)$$

$$-f_k'' + \frac{C''}{C} f_k = k^2 f_k$$

Normalization:

$$f_k f_k^{*\prime} - f_k' f_k^* = \frac{i}{(2\pi)^3}$$

for de Sitter: $f_k = \frac{1}{(2\pi)^{3/2}\sqrt{2|k|}} e^{-i|k|\eta}\left(1 - \frac{i}{|k|\eta}\right)$

for flat: $\tilde{f}_k = \frac{1}{(2\pi)^{3/2}\sqrt{2|k|}} e^{-i|k|\eta}$

$$\beta_{kk'} = i \int d^3x \, \frac{C_0^2}{(2\pi)^3 \, 2\sqrt{|k||k'|}} \, *$$

$$* \left[\frac{e^{-i|k|\eta}}{C(\eta)}\left(1 - \frac{i}{|k|\eta}\right) e^{ikx} \left(\frac{e^{-i|k'|\eta}}{C_0}\right)' e^{ik'x} \right.$$

$$\left. - \left(\frac{e^{-i|k|\eta}}{C(\eta)}\left(1 - \frac{i}{|k|\eta}\right)\right)' e^{ikx} \, \frac{e^{-i|k'|\eta}}{C_0} e^{ik'x} \right] \Bigg|_{\eta=\eta_0}$$

$$= - \frac{i}{2|k|\eta_0} \, \delta(\vec{k} + \vec{k}') \, e^{-2i|k|\eta_0}$$

$$\alpha_{kk'} = -i \int d^3x \, \frac{C_0^2}{(2\pi)^3 2\sqrt{|k||k'|}} \cdot$$

$$\cdot \left[\frac{e^{-i|k|y}}{C(y)} \left(1 - \frac{i}{|k|y}\right) e^{ikx} \left(\frac{e^{i|k|y}}{C_0}\right)' e^{-ik'x} - \right.$$

$$\left. - \left(\frac{e^{-i|k|y}}{C(y)} \left(1 - \frac{i}{|k|y}\right)\right)' e^{ikx} \, \frac{e^{i|k|'y}}{C_0} e^{-ik'x} \right]$$

$$= \left(1 - \frac{i}{2|k|y_0}\right) \delta(\vec{k} - \vec{k}')$$

flat

$$b_{k'} = \int d^3k \left(\alpha_{kk'} a_k + \beta^*_{kk'} a^+_k \right) =$$

$$= \left(1 - \frac{i}{2|k'|y_0}\right) a_{k'} + \frac{i e^{2i|k'|y_0}}{2|k'|y_0} a^+_{-k'}$$

de Sitter

$$\langle b^+_{k_1} b_{k_2} \rangle_{BD} = \frac{1}{4(k_1 y_0)^2} \delta(\vec{k}_1 - \vec{k}_2)$$

$$\Rightarrow n = \int \frac{d^3k}{(2\pi)^3} \frac{1}{4(ky_0)^2} = \int \frac{d^3p}{(2\pi)^3} \frac{H^2}{4p^2}$$

particle density

$$ky_0 = \frac{k}{C_0 H} = \frac{p}{H}$$

For long modes $\frac{H}{p} \gg 1 \Rightarrow$ large occupation numbers

$$\langle b_{k_1} b_{k_2} \rangle_{BD} = e^{2i|k|y_0} \left(\frac{i}{2k_1 y_0} + \frac{1}{4(k_1 y_0)^2} \right) \delta(\vec{k}_1 + \vec{k}_2)$$

$$\langle b^+_{k_1} b^+_{k_2} \rangle_{BD} = e^{-2ik_1 y_0} \left(-\frac{i}{2k_1 y_0} + \frac{1}{4(k_1 y_0)^2} \right) \delta(\vec{k}_1 + \vec{k}_2)$$

$$\langle \varphi^2 \rangle = \frac{1}{C_0^2} \int \frac{d^3k_1\, d^3k_2}{(2\pi)^3\, 2\sqrt{k_1 k_2}} \cdot$$

$$\cdot \langle (e^{-ik_1 y} e^{ik_1 x} b_{k_1} + e^{ik_1 y} e^{-ik_1 x} b^+_{k_1}) \cdot$$

$$\cdot (e^{-ik_2 y} e^{ik_2 x} b_{k_2} + e^{ik_2 y} e^{-ik_2 x} b^+_{k_2}) \rangle_{BD}$$

$$= \frac{1}{C_0^2} \int \frac{d^3k_1\, d^3k_2}{(2\pi)^3\, 2\sqrt{k_1 k_2}} \Big[e^{-i(|k_1|+|k_2|)y} e^{i(k_1+k_2)x} \cdot$$

$$\cdot e^{2i|k_1|y_0} \left(\frac{i}{2k_1 y_0} + \frac{1}{4(k_1 y_0)^2} \right) \delta(\vec{k}_1 + \vec{k}_2) +$$

$$+ e^{i(k_1-k_2)y} e^{i(k_2-k_1)x} \frac{1}{4(k_1 y_0)^2} \delta(\vec{k}_1 - \vec{k}_2) +$$

$$+ e^{-i(k_1-k_2)y} e^{-i(k_2-k_1)x} \left(\frac{1}{4(k_1 y_0)^2} + 1 \right) \delta(\vec{k}_1 - \vec{k}_2) +$$

$$+ e^{i(k_1+k_2)y} e^{-i(k_1+k_2)x} e^{-2ik_1 y_0} \left(-\frac{i}{2k_1 y_0} + \frac{1}{4(k_1 y_0)^2} \right) \cdot$$

$$\cdot \delta(\vec{k}_1 + \vec{k}_2) \Big]$$

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$$\langle\varphi^2\rangle = \frac{1}{C_0^2}\int\frac{d^3k}{(2\pi)^3 2k}\left[1+\frac{1}{2(k\eta_0)^2}+\right.$$

$$\left.+\frac{\cos 2k(\eta-\eta_0)}{2(k\eta_0)^2}+\frac{\sin 2k(\eta-\eta_0)}{k\eta_0}\right]$$

Long wavelengths : $k\eta_0 \ll 1$

$$\Rightarrow \langle\varphi^2\rangle = \frac{1}{C_0^2\eta_0^2}\int\frac{d^3k}{(2\pi)^3 2k^3}\,\frac{1+\cos 2k(\eta-\eta_0)}{2} =$$

$$= \frac{H^2}{4\pi^2}\int\frac{dk}{k}\cos^2 k(\eta-\eta_0)$$

At $\eta=\eta_0$: $\langle\varphi^2\rangle = \frac{H^2}{4\pi^2}\int\frac{dk}{k}$ — amplitude of primordial perturbations

Afterwards $\varphi \sim \frac{H}{2\pi}\cos k(\eta-\eta_0)$ — the phase is also fixed

2) Squeezed states

Ex. 25

Show that

$$a_k = e^{i\alpha}e^{A}b_k e^{-A}$$

where $A = \frac{1}{2}\int dk\left(\zeta(k)\, b_k b_{-k} - \zeta^*(k)\, b_k^+ b_{-k}^+\right)$

$|\zeta| = \operatorname{arcsh}\frac{1}{2k\eta_0}$; $\alpha = -\arcsin\frac{1}{\sqrt{1+4(k\eta_0)^2}}$

$$\arg \zeta = -2ky_0 + \alpha - \frac{\pi}{2}$$

Hint: use

$$e^{A} b_k e^{-A} = b_k + [A, b_k] + \frac{1}{2}[A[A, b_k]] + \dots$$

$$\Rightarrow |0\rangle_{BD} = e^{A}|0\rangle =$$

$$= \exp \int dk \, \frac{1}{2} \left(\zeta(k) b_k b_{-k} - \zeta^*(k) b_k^+ b_{-k}^+ \right) |0\rangle$$

Introduce for $ky_0 \ll 1$:

$$\varphi_k = \frac{b_k + b_{-k}^+}{c_0 \sqrt{2k}} \quad ; \quad p_k = -i c_0 \sqrt{\frac{k}{2}} \left(b_{-k} - b_k^+ \right)$$

Consider the Wigner function:

$$W[\varphi_k, p_k] = \int \frac{d^2 \xi}{(2\pi)^2} \underbrace{\langle \varphi_k - \frac{\xi}{2}|}_{} 0\rangle_{BD} \langle 0 | \varphi_k + \frac{\xi}{2} \rangle e^{-i p_k \xi}$$

$$\hat{\varphi}_k |\varphi_k\rangle = \varphi_k |\varphi_k\rangle$$

Ex. 26

Show that

$$W \propto e^{-\frac{2k^3}{H^2}|\varphi_k|^2} e^{-\frac{H^2}{2k^3}|p_k|^2}$$

$$\delta\varphi_k \delta p_k \sim 1 \qquad \delta\varphi_k \sim \frac{H}{\sqrt{2} k^{3/2}} \to \infty$$

$$\delta p_k \sim \frac{\sqrt{2} k^{3/2}}{H} \to 0$$

3) Classicalization

$$\langle G[\varphi(k)]\, G^+[\varphi(k)]\rangle$$

$$\varphi(k) = \frac{1}{C(\eta)}\left(f_k(\eta)a_k + f^*_{-k}(\eta)\, a^+_{-k}\right)$$

$$G = \sum c_n\, (\varphi(k))^n$$

If f_k is large $\Rightarrow$ $f_k f_k^{*\prime} - f_k' f_k^* = \underbrace{\frac{i}{(2\pi)^3}}_{\text{negligible}}$

$\Rightarrow$ f_k is approximately real

$$\langle G G^+\rangle = \frac{1}{C^2(\eta)} \sum_{n,m} c_n c_m^* \left(f_k(\eta)\right)^{n+m} \cdot$$

$$\cdot \left\langle \left(a_k + a^+_{-k}\right)^n \left(a_k^+ + a_{-k}\right)^m \right\rangle =$$

$$= \frac{1}{C^2(\eta)} \sum_n |c_n|^2 f_k^{2n}(\eta)\, n!$$

This is the same as

$$\int [D\varphi]\, |G(\varphi(k))|^2 \rho(\varphi),$$

where $\rho(\varphi) = \frac{1}{\pi |f_k|^2}\, e^{-\frac{C^2(\eta)\, |\varphi(k)|^2}{|f_k(\eta)|^2}}$

$\Rightarrow$ φ - classical stochastic field

$$\varphi(k,y) = \frac{f_k(y)}{c(y)}\, e(k)$$

$\langle e(k)\, e^*(k') \rangle = \delta(k-k')$ - Gaussian variable

Decoherence occurs when we neglect the decaying mode:

$$f_k = C(y)\left(A_1(k) + A_2(k)\int_0^\infty \frac{dy'}{c^2(y')}\right)$$

This happens shortly after the horizon crossing.