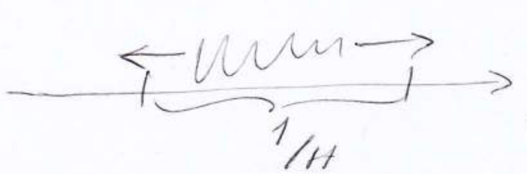
Lecture 9

Stochastic approach to perturbation theory in de Sitter

Ref. Starobinsky, Yokoyama, PRD, 50, 6357 (1994)

1) $ds^2 = -dt^2 + C^2(t)\, d\vec{x}^2$, $C = e^{Ht}$



quantum

classical

$$(*) \quad \varphi(t,x) = \bar{\varphi}(t,x) + \int d^3k\, \theta\left(k - \varepsilon C(t) H\right) \cdot \left(\varphi_k(t) e^{i\vec{k}\vec{x}} a_k + \varphi_k^* e^{-i\vec{k}\vec{x}} a_k^+\right)$$

$\bar{\varphi}$ — averaged over distances $\frac{1}{H\varepsilon}$ – bigger than $\frac{1}{H}$

$$\varphi_k = \frac{1}{(2\pi)^{3/2}} \frac{H}{\sqrt{2k}} \left(\eta - \frac{i}{k}\right) e^{-ik\eta}$$

Add a potential

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0$$

The potential is assumed to be shallow:

$$V'' \ll H^2 \quad (m \ll H^2)$$

$\Rightarrow$ the averaged component evolves slowly

Substitute $(*)$, keep only terms of order ε, neglect $\ddot{\bar{\varphi}}$

$$\Rightarrow \quad \dot{\bar{\varphi}} = -\frac{1}{3H} V'(\bar{\varphi}) + f(t,x)$$

$$f(t,x) = \varepsilon H^2 C(t) \int d^3k \, \delta(|k| - \varepsilon C(t) H) \cdot$$
$$\cdot \left(a_k \varphi_k + a^+_{-k} \varphi^*_{-k} \right) e^{ikx}$$

Because $\varepsilon \ll 1$, f is already given by superhorizon modes $\Rightarrow$ classical noise

<u>Ex. 27</u> Compute

$$\langle f(t_1,x_1) f(t_2,x_2) \rangle = \frac{H^3}{4\pi^2} \delta(t_1 - t_2) j_0 \left(\varepsilon H C(t) |\vec{x}_1 - \vec{x}_2| \right)$$

$$j_0(z) = \frac{\sin z}{z}$$

2) One-point distribution

$$\rho(\varphi): \quad P(\varphi_1 < \bar{\varphi}(t,x) < \varphi_2) = \int_{\varphi_1}^{\varphi_2} d\varphi\, \rho(\varphi)$$

regular motion + diffusion

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \varphi} \mathcal{F}$$

flux $\mathcal{F} = \rho \dot{\varphi} = -\frac{\rho V'}{3H} + \rho f$

$$\frac{\partial \rho}{\partial t} = \underbrace{\frac{1}{3H}\frac{\partial}{\partial \varphi}\left(V'(\varphi)\rho(\varphi)\right) + \frac{H^3}{8\pi^2}\frac{\partial^2 \rho}{\partial \varphi^2}}_{\Gamma_\varphi \rho}$$

Stationary solution (if exists)

$$\frac{\partial \rho}{\partial t} = 0 \to \rho(\varphi) = \mathcal{N} e^{-\frac{8\pi^2}{3H^4} V(\varphi)}$$

Example 1 Massive field

$$V(\varphi) = \frac{m^2 \varphi^2}{2}, \quad m \ll H$$

$$\rho = \mathcal{N} e^{-\frac{4\pi^2}{3H^4} m^2 \varphi^2} \text{ – Gaussian}$$

$$\langle \bar{\varphi}^2 \rangle = \frac{3H^4}{8\pi^2 m^2} \gg \frac{H^2}{4\pi^2}$$

Recall: $\langle \varphi^2 \rangle = \frac{H^2}{4\pi^2} \int \frac{dk}{k} \sim \frac{H^2}{4\pi^2} \ln\left(\frac{k_S}{k_L}\right)$ $\sim \frac{1}{\eta}$

for a massive field:

$$\varphi_k^{(m)} \propto H(-\eta)^{3/2} H_\nu^{(1)}(-k\eta), \quad \nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$$

$$\varphi_k^{(m)} \underset{\eta \to 0}{\propto} H(-\eta)^{3/2} \frac{1}{(-k\eta)^\nu} \propto \frac{H}{k^{3/2}} (-k\eta)^{\frac{m^2}{3H^2}}$$

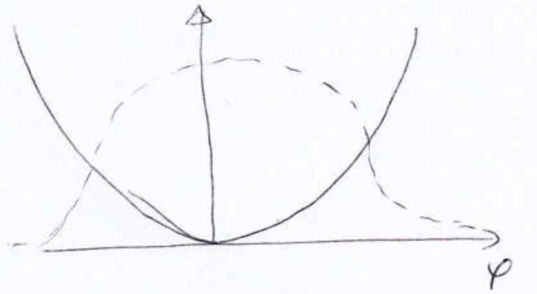
$$\langle \varphi^2 \rangle = \frac{H^2}{4\pi^2} \int \frac{dk}{k} (-k\eta)^{\frac{2m^2}{3H^2}} = \frac{H^2}{4\pi^2} \frac{3H^2}{2m^2} \underbrace{(-\eta k_S)^{\frac{2m^2}{3H^2}}}_{\sim 1}$$

N.B. The modes saturating the integral are exponentially long:

$$k_L \sim \frac{1}{\eta} e^{-\frac{3H^2}{2m^2}} \Rightarrow p_{phys} = H e^{-\frac{3H^2}{2m^2}}$$

Example 2

$$V(\varphi) = \frac{\lambda \varphi^4}{4}$$

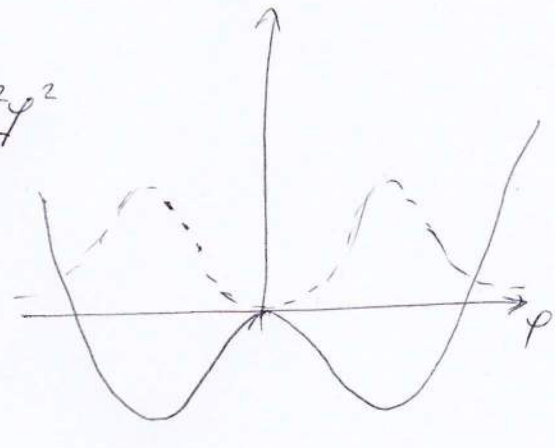


$$\rho = \mathcal{N} e^{-\frac{2\pi^2 \lambda}{3H^4}\varphi^4} \quad \text{- non Gaussian}$$

$$\langle \varphi^2 \rangle \sim \frac{H^2}{\sqrt{\lambda}} \cdot (0.1)$$

Example 3

Double-well: $V = \frac{\lambda \varphi^4}{4} - \frac{m^2 \varphi^2}{2}$



Time dependence

Ansatz: $\rho = e^{-v(\varphi)} \sum_{n=0}^{\infty} a_n \Phi_n(\varphi)\, e^{-\Lambda_n (t-t_0)}$

$$v(\varphi) = \frac{4\pi^2}{3H^4} V(\varphi)$$

eigenvalue equation $\Rightarrow$ $\left(-\frac{1}{2}\frac{\partial^2}{\partial \varphi^2} + W(\varphi)\right)\Phi_n(\varphi) = \frac{4\pi^2 \Lambda_n}{H^3}\Phi_n(\varphi)$

$$W(\varphi) = \frac{1}{2}\left((v'(\varphi))^2 - v''(\varphi)\right)$$

$$a_n = \int d\varphi\, e^{v(\varphi)} \rho(\varphi, t_0)\, \Phi_n(\varphi)$$

If stationary state exists $\Rightarrow \Lambda_0 = 0$
Convergence to the stationary state is determined by Λ_1

$$V = \frac{m^2\varphi^2}{2} \Rightarrow W(\varphi) \propto \frac{m^4}{H^8}\varphi^2 \Rightarrow \Lambda_1 \propto \frac{m^2}{H}$$

$$V = \frac{\lambda\varphi^4}{4} \Rightarrow W(\varphi) = \frac{1}{2}\left[\left(\frac{4\pi^2}{3H^4}\right)^2 \lambda^2\varphi^6 - \frac{4\pi^2}{3H^4}\, 3\lambda\varphi^2\right] \sim$$

$$\sim \frac{\lambda^2}{H^8}\cdot\left(\frac{H}{\lambda^{1/4}}\right)^6 \sim \frac{\sqrt{\lambda}}{H^2} \Rightarrow \Lambda_1 \sim H\sqrt{\lambda}$$

Ex. 28 For $V = -\frac{m^2\varphi^2}{2} + \frac{\lambda\varphi^4}{4}$ show that

$$\Lambda_1 = \frac{\sqrt{2}\, m^2}{3\pi H}\, e^{-\frac{2\pi^2 m^4}{3\lambda H^4}}$$

Ex. 29 Consider free massless scalar field
Find the time-dependence of ρ, $\langle\varphi^2\rangle$.
Take initial condition

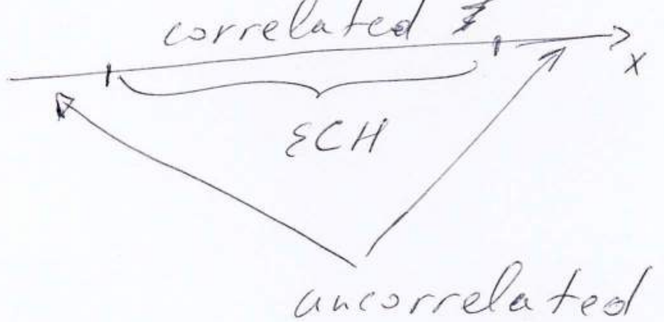
$$\rho(t=0) = \delta(\varphi)$$

3) Two-point correlations

$$\frac{\partial \rho^{(2)}}{\partial t}\left[\varphi_1(t, x_1), \varphi_2(t, x_2)\right] = \Gamma_{\varphi_1} \rho^{(2)} + \Gamma_{\varphi_2} \rho^{(2)} +$$

$$+ j_0\left(\varepsilon C(t) H |x_1 - x_2|\right) \frac{H^3}{4\pi^2} \frac{\partial^2 \rho^{(2)}}{\partial \varphi_1 \partial \varphi_2}$$

approximate $j_0(x) \simeq \theta(1-x)$



If $j_0 = 1$ (full correlation)

$\Rightarrow \rho^{(2)} = \rho_{eq}(\varphi_1)\, \delta(\varphi_1 - \varphi_2)$ is a solution

valid for $t < -\frac{1}{H} \ln\left(\varepsilon H |x_1 - x_2|\right)$

for $t > -\frac{1}{H} \ln\left(\varepsilon H |x_1 - x_2|\right)$

$$\Rightarrow \frac{\partial \rho^{(2)}}{\partial t} = \Gamma_{\varphi_1} \rho^{(2)} + \Gamma_{\varphi_2} \rho^{(2)}$$

(8)

Solve using the Green's function $\Pi(\varphi_1, \varphi_0; t_1, t_0)$:

$$\begin{cases} \dfrac{\partial \Pi}{\partial t_1} = \Gamma_{\varphi_1} \Pi \\ \Pi \big|_{t_1 = t_0} = \delta(\varphi_1 - \varphi_0) \end{cases}$$

$$\Pi = e^{-v(\varphi_1)} \sum_n \Phi_n(\varphi_1) \Phi_n(\varphi_0) e^{-\Lambda_n (t_1 - t_0)} e^{v(\varphi_0)}$$

$$\rho^{(2)} = \int d\varphi \, \rho_{1eq}(\varphi) \, \Pi(\varphi_1, \varphi; t, t_r) \, \Pi(\varphi_2, \varphi; t, t_r)$$

$$t_r = -\frac{1}{H} \ln(\varepsilon H r)$$

$$G(x_1, x_2; t) \equiv \langle \varphi(x_1, t) \varphi(x_2, t) \rangle =$$

$$= \int d\varphi_1 d\varphi_2 \, \rho(\varphi_1, \varphi_2; t) \varphi_1 \varphi_2 =$$

$$= \int d\varphi_1 d\varphi_2 \, \varphi_1 \varphi_2 \, e^{-v(\varphi_1) - v(\varphi_2)}$$

$$\int d\varphi_0 \sum_{n,m} \Phi_n(\varphi_1) \Phi_n(\varphi_0) \Phi_m(\varphi_2) \Phi_m(\varphi_0) \cdot$$

$$\cdot \underbrace{e^{2v(\varphi_0)} \rho_{eq}(\varphi_0)}_{\text{const}} e^{-\Lambda_n (t - t_r) - \Lambda_m (t - t_r)}$$

$$= \int d\varphi_1 d\varphi_2 \, \varphi_1 \varphi_2 \, e^{-v(\varphi_1) - v(\varphi_2)} \sum_n \Phi_n(\varphi_1) \Phi_n(\varphi_2) \cdot$$

$$\cdot e^{-2\Lambda_n (t - t_r)}$$

$$G(x_1, x_2; t) \propto e^{-2\Lambda_1 (t - t_r)} =$$

$$= e^{-2\Lambda_1 \left(t + \frac{1}{H} \ln(\varepsilon H r)\right)} =$$

$$= e^{-2\Lambda_1 \left(t + \frac{1}{H} \ln Hr\right)} e^{-\frac{2\Lambda_1}{H} \ln \varepsilon}$$

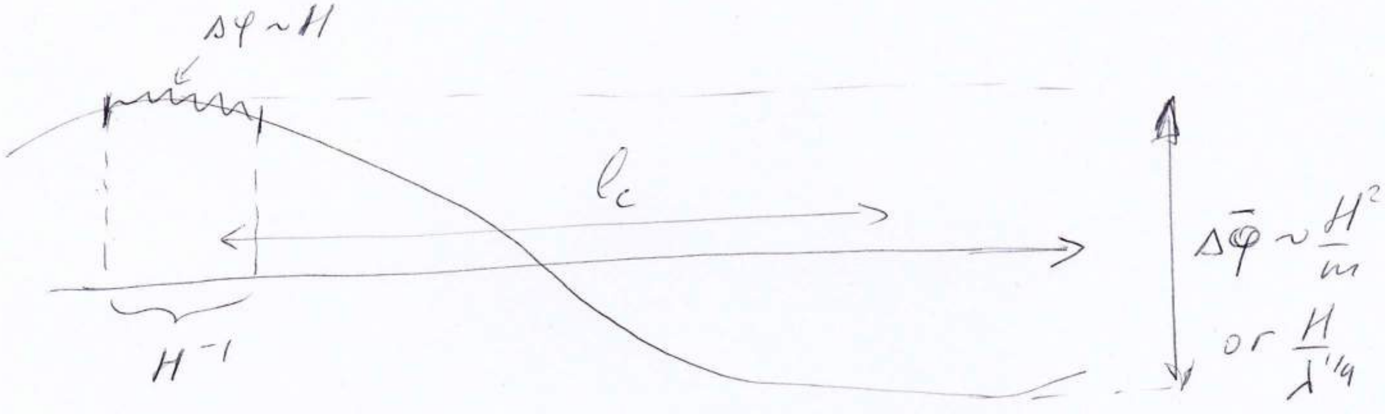
if $-\frac{\Lambda_1}{H} \ln \varepsilon \ll 1 \Leftrightarrow \varepsilon \gg e^{-\frac{H}{\Lambda_1}} \Rightarrow$

$$\Rightarrow G(x_1, x_2, t) \sim e^{-2\Lambda_1 \left(t + \frac{1}{H} \ln Hr\right)}$$

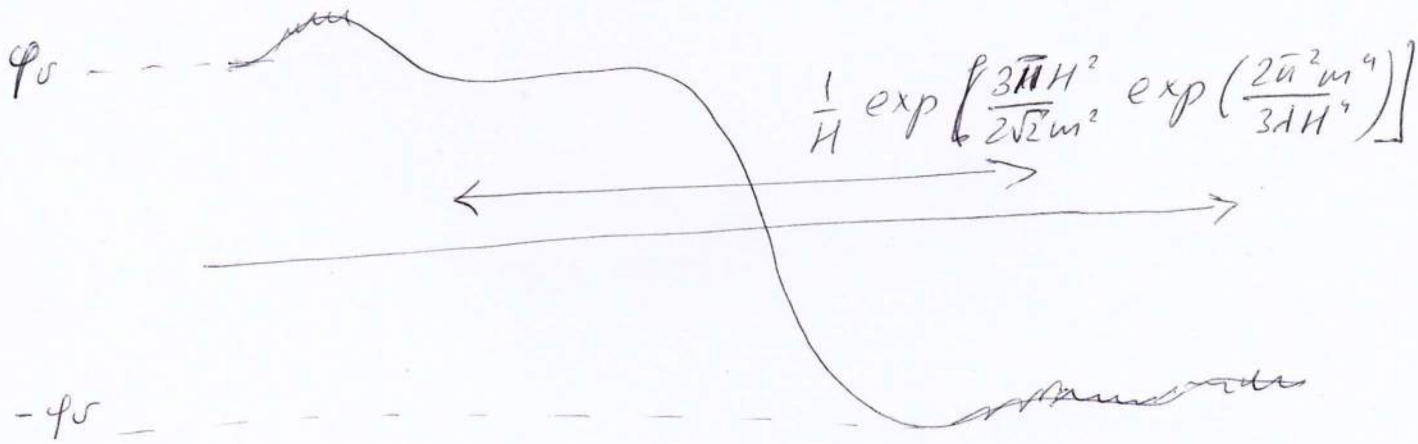
$$l_{ph} = r e^{Ht} \Rightarrow \underbrace{G \propto e^{-\frac{2\Lambda_1}{H} \ln (H l_{ph})}}$$

the Green's function is stationary in <u>physical</u> distance (self-~~re~~producing evolution)

$l_c = \frac{1}{H} e^{\frac{H}{2\Lambda_1}}$ – exponentially large correlation length



For double-well => third correlation scale



Derivation is applicable if $t_{inf} > t_c = \frac{\cancel{H}\,1}{\Lambda_1}$