Dealer Intermediation in OTC Markets with Private Valuation*

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Abstract

We develop a general equilibrium theory of dealer intermediation in over-the-counter (OTC) markets with private investor valuations. Trading occurs through a competitive request-for-quote (RFQ) protocol in which dealers submit price quotes without observing investor types. The model nests the standard voice trading channel. We prove the existence and uniqueness of a stationary equilibrium and characterize it through a system of functional equations. We decompose the effects of adverse selection into a scale effect, reflecting the mass of investors on each side of the marginal type, and a composition effect, reflecting how private information shapes the quality of trading opportunities. This decomposition determines how asymmetries between the bid and ask sides of the market affect price elasticities. In equilibrium, quote distributions generate bid–ask spreads, price dispersion and strictly positive probabilities of trade failure. The model provides a tractable framework to study how dealer competition, supply, investor heterogeneity and search frictions jointly determine liquidity, allocation and welfare.

JEL classification: D82, D53, D83, G14

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1 Introduction

Dealer intermediation plays a central role in many over-the-counter (OTC) markets. In these environments, investors typically trade with dealers in decentralized transactions, while dealers transact with one another in interdealer markets to manage inventory and risk. Theoretical search models of dealer intermediated OTC markets have almost exclusively focused on settings in which dealers and investors share symmetric information about each other's characteristics. Following the seminal framework of Duffie, Gârleanu and Pedersen (2005) (DGP), the standard approach is to set prices through bargaining mechanisms that allocate surplus between trading parties[1].

While this approach is highly tractable and greatly simplifies the characterization of the equilibrium, it has important limitations. First, it abstracts from adverse selection, as investors have incentives to misrepresent their private valuations. Moreover, bid–ask spreads in these models often arise from discrete heterogeneity and tend to vanish when investor types are modeled as a continuum. Finally, bargaining frameworks rule out trade failure by construction and imply that a transaction always occurs upon dealer-investor meetings whenever there is positive surplus. This last point is particularly at odds with recent empirical evidence about markets organized around the request-for-quotes (RFQ) protocol.

In RFQ trading, investors submit trade requests to an electronic platform. After observing a request, competing dealers have a limited time to submit price quotes. The investor then observes the set of quotes and decides whether to execute the trade. These auction-based protocols are increasingly prevalent in markets such as corporate bonds, interest rate swaps, and credit default swaps. RFQ markets are particularly well suited for studying price formation under adverse selection, in contrast to standard bargaining approaches. Indeed, prices are submitted without direct negotiation between investors and dealers, which departs from the traditional voice trading view of OTC markets. Moreover, recent empirical studies document that, under RFQ (and similar) trading arrangements, investors sometimes decline the quotes they receive. Examples include Hendershott and Madhavan (2015) and Kargar, Lester, Plante and Weill (2023) for corporate bonds, Riggs, Onur, Reiffen and Zhu (2020) for credit default swaps, and Hendershott, Li, Livdan and Schürhoff (2024) for collateralized loan obli-

[1] Other examples of OTC models with bargaining include Duffie, Gârleanu and Pedersen (2007); Vayanos and Weill (2008); Lagos and Rocheteau (2009); Lagos, Rocheteau and Weill (2011); Üslü (2019); Hugonnier, Lester and Weill (2020). A broader literature review is provided in Hugonnier, Lester and Weill (2025)

gations. This pattern strongly suggests that dealers are imperfectly informed about the valuations of investors, as otherwise they would not submit quotes that are rejected.

Motivated by these observations, we develop a new theory of dealer intermediation in OTC markets with search frictions. In our model, dealers do not observe the private asset valuations of investors at the time of trade. Transactions occur through an RFQ auction, although the framework also nests the standard voice trading environment. In equilibrium, dealers draw prices from a bid and ask price distributions. Prices, valuations, and allocations are endogenously determined by supply and demand in the asset market. Combining adverse selection with general equilibrium feedback generates asymmetric responses from the bid and ask sides of the market when trading conditions change. It also overturns standard intuitions from static adverse selection frameworks, in particular regarding the effects of competition. At the same time, the model preserves key features of these frameworks, such as bid–ask spreads and the possibility of market failure, while endogenizing the probability of trade.

We decompose the effects of adverse selection into a scale effect and a composition effect. The scale effect captures the mechanical relationship between the marginal type and the mass of investors on each side of the market, that is, those willing to sell and those willing to buy. In particular, a lower marginal valuation mechanically leads to lower bid and ask prices. The composition effect is more subtle and captures how changes in the marginal type reshape the distribution of investor types on each side of the market. Beyond the change in mass, the relative composition of buyers and sellers is altered, which affects the quality of trading opportunities faced by dealers. As a result, after controlling for scale by rescaling distributions to a common support, changes in the marginal type can improve or deteriorate the effective quality of price distributions depending on the underlying distribution of valuations.

We show that composition effects can either amplify or dampen price elasticities relative to the standard benchmark without private information, where only scale effects operate. Moreover, a similar decomposition arises when studying bid–ask spreads and the mass of excluded investors. In particular, trade probabilities and markups are entirely driven by composition effects. While it is well understood that adverse selection and the shape of the distribution of heterogeneity matter for pricing frictions[2], our contribution is to endogenize this shape through general equilibrium

[2]Standard references include Myerson and Satterthwaite (1983); Wilson (1985); Glosten and Milgrom (1985); Glosten (1989).

effects. In particular, the marginal type determines how investors are split between the bid and ask sides of the market, which characterizes the effective distribution of adverse selection faced by dealers. This allows us to study how primitive parameters such as supply, preferences, and search frictions shape market failure and liquidity in OTC markets.

A separate implication of the model concerns the effects of dealer competition at the auction stage of an RFQ. In particular, the general equilibrium feedback on asset allocations and valuations leads to a surprising outcome. In contrast to a static RFQ auction, where valuations and allocations are exogenous, we find that prices, when expressed in type space, are not necessarily more favorable as competition increases. Indeed, greater competition makes the distribution of asset holdings less concentrated around the marginal type, which induces dealers to offer lower bids and higher asks with positive probability. As a result, investors who would have traded at every opportunity under lower competition may optimally refuse to trade. Overall, increasing competition can expand price dispersion at both ends of the price distributions.

The model builds on the seminal framework of Duffie et al. (2005). Investors who wish to trade the asset face search frictions and can only transact through dealers, whom they meet randomly over time. Investors also experience preference shocks and draw new utility flows from an exogenous distribution. Dealers trade in a frictionless interdealer market and immediately clear their inventory after trading with investors. The equilibrium interdealer price adjusts to clear the market. We depart from the standard framework in three ways. First, we assume that investors have a continuous distribution of types, following Hugonnier, Lester and Weill (2022). Second, trades occur through the RFQ protocol, in which dealers compete in an auction when submitting price quotes to an investor, similar to Weill (2020). Finally, we assume that investor types are private information and are not observed by dealers, so that dealers cannot condition their quotes on the preference parameter of investors.

We first analyze the static RFQ auction faced by dealers, by fixing an exogenous distribution of reservation prices on the bid and ask sides of the market and an interdealer price. The auction extends Burdett and Judd (1983) to a framework with private valuations. We characterize the unique Bayesian equilibrium of the auction in closed-form. Its properties are key to building intuition for the scale and composition effects of adverse selection, as well as for the effects of competition, when we turn to the dynamic general equilibrium OTC market model.

In the dynamic model, reservation values, asset allocations across types, bid and ask price distributions, and the interdealer price are jointly determined in equilibrium to clear the market. We show that a unique stationary equilibrium exists. Existence is established by proving that a fixed point of an operator jointly determining price distributions and the marginal type exists, in the spirit of general equilibrium theory. Uniqueness is obtained by deriving necessary and sufficient conditions for the equilibrium, and reducing the infinite-dimensional problem to a two-dimensional one with monotonic properties. This construction allows us to characterize the equilibrium precisely, either through a system of Riccati equations or via a numerical method for which we establish convergence. We believe this level of tractability is new in the literature on OTC markets with private information.

When investors are uniformly distributed and trades occur through the standard voice channel, while maintaining private information, the equilibrium admits an explicit characterization. Specifically, reservation values, asset allocations across types, bid and ask price distributions, and the interdealer price can all be expressed in closed-form. We use this explicit characterization to confirm our results on the scale and composition effects of adverse selection. In particular, since the shape of the uniform distribution is invariant to the choice of the cutoff, the bid and ask sides of the market are symmetric in composition, and the marginal type coincides with the benchmark without private information. Although the equilibrium quote distributions generate price dispersion, their supports remain small for empirically plausible parameter values.

Literature review Our paper relates to several strands of the literature on price formation and OTC markets. In our model, competing dealers draw prices from a distribution in a mechanism similar to Burdett and Judd (1983). However, our auction features asymmetric information about the private valuations of investors, as in Garrett, Gomes and Maestri (2019). In contrast to both of these papers, the payoff of the dealer is not monotonic in the quoted price. Indeed, offering the lowest possible bid yields zero expected profit because the investor never accepts the trade, while offering the highest possible bid, which is the interdealer price, yields zero realized profit. We extend the analysis to accommodate this non-monotonicity by applying an ironing procedure in the spirit of Myerson (1981) and Rochet and Choné (1998). Our mechanism is also related to Biais, Martimort and Rochet (2000), who study competition among dealers in the presence of adverse selection, although in a different centralized setting. Their paper also features asymmetries between the bid and ask sides of the market, but they do not separate the scale and composition effects, as the dealer valuation of the asset (the analogous to our interdealer price) is quasi-exogenous. In these papers, the utility of agents and the distribution of types are taken as exogenous, or at least independent of the principals' actions. In contrast, in our model, both the utility of agents and the distribution of types over which screening occurs are affected by dealers' actions in a dynamic general equilibrium environment with search frictions.

Our paper also contributes to the theoretical literature on RFQ markets. Weill (2020) studies a model related to ours, except that dealers know investors' valuations of the asset. The model provides a microfoundation for the bargaining parameter in Duffie et al. (2005) and is qualitatively equivalent. Closely related mechanisms are studied in Duffie, Dworczak and Zhu (2017), Vogel (2019), and Maciocco (2025). Another strand of the RFQ literature studies a single auction event. In particular, Glebkin, Yueshen and Shen (2023), Wang (2023), and Baldauf and Mollner (2024) analyze settings in which investors optimally choose to contact a finite number of dealers. Investors may limit the number of contacted dealers to maintain relationships or to prevent the leakage of their trading motives. We abstract from these considerations and instead take the distribution of the number of responding dealers as exogenous. Zhu (2012) studies a model in which a single investor has private information and dealers compete to trade with her. In contrast to RFQ trading, dealers submit quotes sequentially rather than simultaneously. None of these papers study a setting with privately informed investors in a general equilibrium framework.

A small literature studies dynamic OTC markets in the presence of asymmetric information. Chiu and Koeppl (2016) analyze a model with two types of assets, good and bad, in which sellers have private information about the quality of the asset. In contrast, in our model, asymmetric information concerns private valuations rather than the common value of the asset. Trading in their model is bilateral and does not involve dealer intermediation. As a result, the inefficiencies they derive differ from ours, which arise in a setting where dealers possess some market power. Bethune, Sultanum and Trachter (2022) consider a model in which investors have private information about their valuation of the asset, similar to ours. However, trading in their model is bilateral and does not involve dealer intermediation. Specifically, upon each meeting either the buyer or the seller makes a take-it-or-leave-it offer. While their model retains the endogeneity of equilibrium objects, it is less tractable: the authors do not establish uniqueness of the equilibrium nor provide a full characterization. Moreover, their framework does not generate the types of inefficiencies we study, such as bid–ask spreads and no-trade regions. Zhang (2018) studies a model close to our nested voice trading version. However, the author additionally assumes that the distribution of utility flows is symmetric around the asset supply $s = 1/2$. This restriction rules out bid–ask asymmetries and prevents the separation between scale and composition effects.

A key methodological difference with our paper, is that, instead of assuming that one party quotes a price upon meeting, Bethune et al. (2022) and Zhang (2018) both adopt a mechanism design approach. Specifically, these authors assume that one of the parties offers a menu that specifies a price and a probability of execution. Under this approach, the terms of trade are known in advance and uncertainty arises only from trade execution. By contrast, in our formulation the price itself is random, and this randomness generates uncertainty over the trade execution. The fact that investors have quasilinear preferences implies that the two approaches are equivalent at the aggregate level despite this difference, as noted by Zhang (2018). Our formulation using quote distributions provides a more tractable approach and extends naturally to RFQ protocols.

Some elements of our framework are related to the results presented in Chapter 10 of Hugonnier et al. (2025), which studies a voice trading environment with private information. In particular, similar necessary and sufficient conditions arise and a closed-form solution under a uniform distribution is available. However, their analysis does not establish the existence or uniqueness of the equilibrium. We also introduce the RFQ trading protocol into the model. This richer framework allows us to derive new economic implications, including the decomposition of adverse selection into scale and composition effects, as well as the analysis of competition in equilibrium. Private valuations are particularly relevant in RFQ markets rather than in voice trading environments, since RFQ trading typically involves limited direct communication between investors and dealers, making the transmission of information less likely.

The remainder of the paper is organized as follows. Section 2 presents the static RFQ auction and derives key properties. Section 3 introduces the dynamic model with search frictions. Section 4 defines the market clearing equilibrium, provides a partial characterization of equilibrium objects, and establishes existence. Section 5 derives necessary and sufficient conditions, proves uniqueness, and presents methods to solve the equilibrium, both in differential form and numerically. Section 6 discusses the economic implications of the model, and section 7 presents a quantitative analysis. Section 8 concludes. Proofs and additional results are provided in the appendix.

2 RFQ auction with private information

In this section, we analyze the pricing problem faced by dealers in RFQ trading. We consider a static environment in which a privately informed investor requests quotes from competing dealers, who submit prices without observing her type. We characterize the equilibrium distribution of quotes for given distributions of reservation prices. This result extends the pricing mechanism of Burdett and Judd (1983) to an environment with private information and will serve as a key building block for the general equilibrium analysis.

2.1 Environment and equilibrium

An investor is characterized by her reservation price $r \in [\underline{r}, \bar{r}] = \mathcal{R}$, which is the minimum price at which she is willing to sell the asset or the maximum price at which she is willing to buy the asset. r is private information of the investor but dealers know the distribution $\Phi_1(r)$ of investors who wish to sell the asset and the distribution $\Phi_0(r)$ of investors who wish to buy the asset. Without loss of generality, we assume that $\underline{r} = \inf \operatorname{supp}(\Phi_0) = \inf \operatorname{supp}(\Phi_1) < \bar{r} = \sup \operatorname{supp}(\Phi_0) = \sup \operatorname{supp}(\Phi_1)$. We consider an investor who submits an RFQ to execute either a buy or sell order. We treat both cases simultaneously in this section, although they can be analyzed separately.

Dealers observe the RFQ and decide whether to submit a quote. We assume that all dealers are symmetric and interchangeable. We take participation in the auction as exogenous and assume that the number of participating dealers, denoted by Ψ, is a random variable with values on the set of strictly positive integers. The only assumption we impose on Ψ is that $\sum_{N=1}^{\infty} \psi_N N < \infty$. Dealers who participate in the auction do not observe how many other dealers participate. From the perspective of a dealer who responds to the RFQ, the number of competing dealers follows a size-biased distribution. In particular, the probability that there are $k-1$ competitors is

$$\nu_k = \frac{\psi_k k}{\sum_{N=1}^{\infty} \psi_N N}.$$

Equivalently, the number of competitors is distributed as $\tilde{\Psi} - 1$, where $\tilde{\Psi}$ is the size-biased distribution[3]. We define $h : [0,1] \to [\nu_1, 1]$

$$h(q) \equiv \sum_{k=1}^{\infty} \nu_k q^{k-1},$$

which is the probability generating function of $\tilde{\Psi} - 1$.

Participating dealers submit quotes, and the investor selects the most favorable one, that is, the highest bid for a sell order or the lowest ask for a buy order. The investor executes the trade if and only if the highest bid exceeds her reservation price r or the lowest ask falls below r. If a dealer's bid is executed, the dealer immediately resells the asset at price P and if a dealer's ask is executed, the dealer immediately buys the asset at price P. We assume that $P \in (\underline{r}, \overline{r})$, which is a natural assumption for dealers to have incentives to quote both bid and ask prices.

We seek for a symmetric Bayesian equilibrium in mixed strategies. Accordingly, we assume that dealers draw bid prices from an endogenous distribution B and ask prices from a distribution A. In this setting, the expected profit of a dealer who submits a bid price b is

$$\begin{aligned}\Pi_b(b|B) \equiv\ & (P-b)\Phi_1(b) \\ & \times \sum_{k=1}^{\infty} \nu_k \sum_{m=0}^{k-1} \frac{1}{1+m} \binom{k-1}{m} (B(b) - B(b^-))^m B(b^-)^{k-1-m}\end{aligned} \tag{1}$$

and the expected profit of a dealer who submits an ask price a is

$$\begin{aligned}\Pi_a(a|A) \equiv\ & (a-P)\left(1 - \Phi_0(a^-)\right) \\ & \times \sum_{k=1}^{\infty} \nu_k \sum_{m=0}^{k-1} \frac{1}{1+m} \binom{k-1}{m} (A(a) - A(a^-))^m (1 - A(a))^{k-1-m}.\end{aligned} \tag{2}$$

The first term in (1) and (2) is the realized profit, the second term is the probability that the investor accepts the quote, and the last term is the probability of winning the auction. In particular, the third term captures the fact that, if m dealers submit the same quote, the investor selects among them uniformly at random.

[3] A simple intuition is $\nu_k = \mathbf{P}(\Psi = k|\text{ the dealer responds}) = \frac{\mathbf{P}(\Psi=k)\mathbf{P}(\text{the dealer responds }|\Psi=k)}{\mathbf{P}(\text{the dealer responds})} = \frac{\psi_k k/\overline{N}}{\sum_{n=1}^{\infty} \psi_N N/\overline{N}} = \frac{\psi_k k}{\sum_{N=1}^{\infty} \psi_N N}$, where $\overline{N}$ is the "total number" of dealers.

Definition 1. *B and A form a symmetric Bayesian equilibrium if and only if*

$$supp(B) \subseteq \arg\max_{b\in[\underline{r},P]} \Pi_b(b|B) \tag{3}$$
$$supp(A) \subseteq \arg\max_{a\in[P,\overline{r}]} \Pi_a(a|A).$$

Lemma 1. *In equilibrium:*

- *If* $\psi_1 = 0$, $B(x) = A(x) = \mathbf{1}_{x\geq P}$.
- *If* $P = \underline{r}$, *then* $B(x) = \mathbf{1}_{x\geq P}$. *If* $P = \overline{r}$, *then* $A(x) = \mathbf{1}_{x\geq P}$.
- *If* $\psi_1 \in (0,1)$ *and* $P \in (\underline{r}, \overline{r})$, *B and A are continuous.*
- *If* $\psi_1 = 1$, $supp(B) \subseteq \text{argmax}_{b\in[\underline{r},P]}(P-b)\Phi_1(b)$
  and $supp(A) \subseteq \text{argmax}_{a\in[P,\overline{r}]}(a-P)\,(1-\Phi_0(a^-))$.

The intuition behind lemma 1 is straightforward. If $\psi_1 = 1$, the sums in (1) and (2) disappear and profits do not depend on B or A. The effects of competition are absent and any pair of distributions supported on profit maximizers forms an equilibrium. If $\psi_1 = 0$, dealers always face at least one competitor, and Bertrand competition drives profits to zero, which corresponds to quoting P. Now consider $\psi_1 \in (0,1)$. If B has a point mass at $b \in [\underline{r}, P)$, dealers can profitably deviate by bidding a small amount $\epsilon > 0$ above b, thereby significantly increasing the probability of winning while preserving a strictly positive margin. This argument rules out atoms in equilibrium.

An important implication of lemma 1 is that the expression of the profit functions (1) and (2) simplify to

$$\Pi_b(b|B) = (P-b)\Phi_1(b)h(B(b)) \tag{4}$$

and

$$\Pi_a(a|A) = (a-P)\left(1-\Phi_0(a^-)\right)h(1-A(a)). \tag{5}$$

The simplification follows because all configurations involving discontinuities in B or A either yield zero profits or correspond to the case of no competition, where $h(q) = 1$.

For the remainder of the section, we focus on the case $\psi_1 \in (0,1)$ as the boundary cases have already been characterized. In this region, the auction equilibrium is unique and admits a closed-form characterization. We define the *monopolistic* profit functions of dealers for a request to sell

$$\hat{\Pi}_b(b) \equiv (P-b)\Phi_1(b)$$

and for a request to buy

$$\hat{\Pi}_a(a) \equiv (a-P)\left(1-\Phi_0(a^-)\right).$$

These functions represent the expected profit of a dealer conditional on being the sole respondent to the RFQ. We also define the right and left running maxima

$$\hat{\Pi}^*_b(b) \equiv \sup_{x\in[b,\bar{r}]} \hat{\Pi}_b(x) \quad \text{and} \quad \hat{\Pi}^*_a(a) \equiv \sup_{x\in[\underline{r},a]} \hat{\Pi}_a(x),$$

as well as the auxiliary function

$$\tilde{\mathbf{h}}(x,y) \equiv h^{-1}\left(\nu_1 \frac{x}{y}\right) \mathbf{1}_{x\geq y\geq \nu_1 x} + \mathbf{1}_{y<\nu_1 x}. \tag{6}$$

Proposition 1. *If* $\psi_1 \in (0,1)$ *and* $P > \underline{r}$*,*

$$B(b) = \tilde{\mathbf{h}}\left(\hat{\Pi}^*_b(\underline{r}), \hat{\Pi}^*_b(b)\right).$$

If $\psi_1 \in (0,1)$ *and* $P < \bar{r}$*,*

$$A(a) = 1 - \tilde{\mathbf{h}}\left(\hat{\Pi}^*_a(\bar{r}), \hat{\Pi}^*_a(a)\right)$$

In particular, $supp(B) \subset [\underline{r}, P)$ *and* $supp(A) \subset (P, \bar{r}]$*.*

The technical details of the proof are provided in appendix A.1. We outline here the main steps of the argument for the bid distribution. In equilibrium, profits must be equalized across all bids in the support so that

$$\Pi_b(b|B) = \hat{\Pi}_b(b) h(B(b)) = \pi^*_b$$

for all $b \in$ supp(B), where $\pi^*_b = \sup_{b\in\mathcal{R}} \Pi_b(b|B)$. Inverting this condition yields

$$B(b) = h^{-1}(\pi^*_b / \hat{\Pi}_b(b))$$

for all $b \in$ supp(B). To ensure that B is increasing, we apply an ironing procedure and replace $\hat{\Pi}_b(b)$ by $\hat{\Pi}^*_b(b)$. The equilibrium profit level is then determined by the boundary condition

$$\pi^*_b = h(0)\hat{\Pi}^*_b(\underline{r}) = \nu_1\hat{\Pi}^*_b(\underline{r}).$$

The end point of the support of B is determined when $\pi^*_b = \hat{\Pi}^*_b(b)$, where $B(b) = 1$. The intuition is summarized in figure 1.

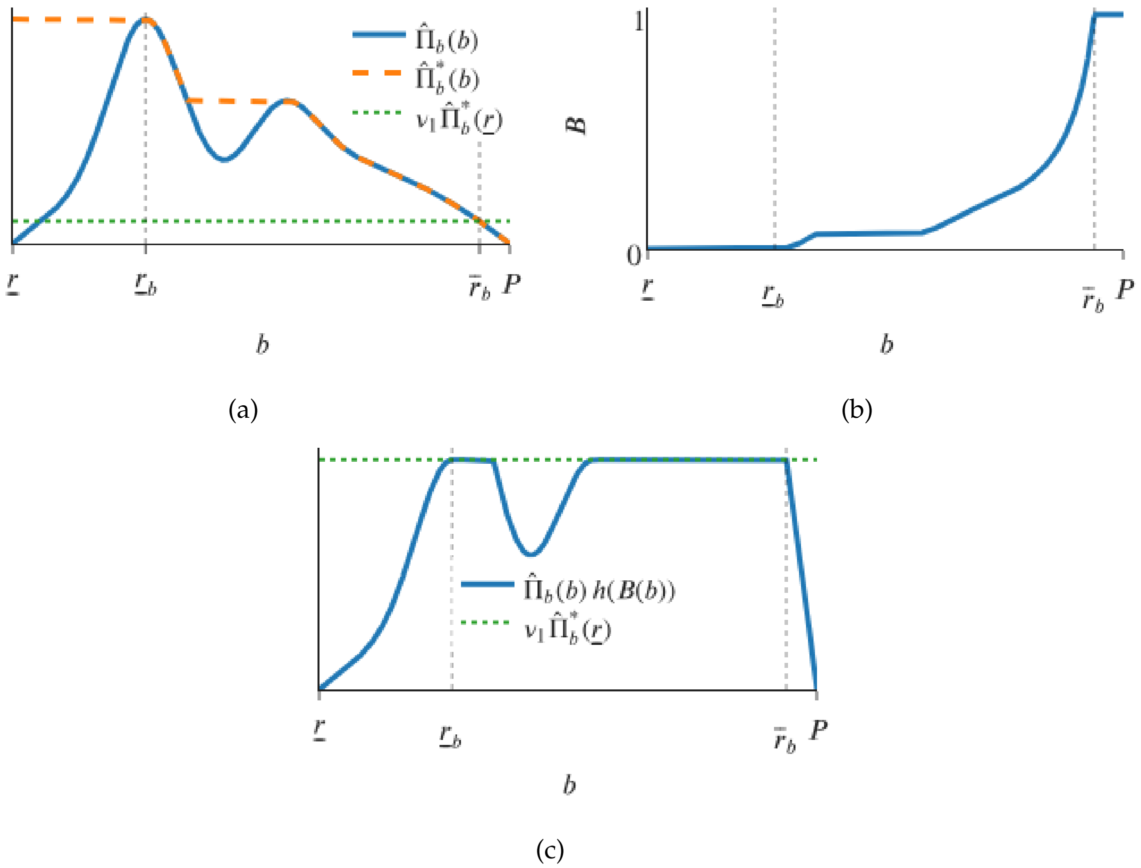

Figure 1: Illustration of the auction solution in case of a request to sell. Panel (a) displays the monopolistic profit function and its ironed counterpart. Panel (b) shows the construction of the equilibrium distribution B. Panel (c) verifies that substituting B into the profit maximizes the profit over its support, consistent with the equilibrium condition.

2.2 Equilibrium structure

The equilibrium solution characterized in proposition 1 exhibits several important properties that are useful for building intuition before turning to the dynamic general equilibrium model. While these properties are derived in a static RFQ setting with multiple competing dealers ($\psi_1 \in (0,1)$), they remain relevant in the dynamic model even in the monopoly case ($\psi_1 = 1$), which can be interpreted as the limit of the RFQ mechanism as the number of responding dealers converges to one. We first denote by $\underline{r}_b, \bar{r}_b$ and $\underline{r}_a, \bar{r}_a$ the boundaries of the supports of B and A respectively,

$$\underline{r}_b \equiv \inf \operatorname{supp}(B), \quad \bar{r}_a \equiv \sup \operatorname{supp}(B), \quad \underline{r}_a \equiv \inf \operatorname{supp}(A), \quad \bar{r}_a \equiv \sup \operatorname{supp}(A).$$

Corollary 1. $\underline{r} \leq \underline{r}_b < \bar{r}_b < P < \underline{r}_a < \bar{r}_a \leq \bar{r}$. *If Φ_1 and Φ_0 are continuous, the inequalities are strict.*

We observe that the static auction generates price dispersion, bid-ask spreads, and a strictly positive probability of trade failure. The bid-ask spread, measured by $\underline{r}_a - \bar{r}_b$, is a standard implication of adverse selection. There is also market failure: investors with $r \in (\bar{r}_b, P)$ and $r \in (P, \underline{r}_a)$ do not trade, although there is positive surplus to be gained for both dealers and investors. In addition, investors with $r \in (\underline{r}_b, \bar{r}_b)$ may reject a bid if the realized best quote is below r, and similarly investors with $r \in (\underline{r}_a, \bar{r}_a)$ may reject an ask if the quote exceeds r. By contrast, for continuous distributions of reservation prices, investors with $r < \underline{r}_b$ always accept the bid they receive, while investors with $r > \bar{r}_a$ always accept the ask. Our next propositions states that if Φ_1 and Φ_0 have log-concave densities, B and A do not have any ironing.

Lemma 2. *If Φ_1 and Φ_0 admit log-concave densities, B and A have connected supports and there is no ironing.*

Log-concavity is a standard assumption in economics and is satisfied by many commonly used distributions, such as the uniform, normal, exponential, and beta distributions, as well as their truncations. It is also well known in mechanism design that log-concavity rules out ironing and bunching.

We now turn to how the interdealer price P affects the auction equilibrium. This will be particularly relevant in the general equilibrium model, where P is determined endogenously. Specifically, we define $B(\cdot \mid P)$ and $A(\cdot \mid P)$ as the bid and ask distributions associated with a given interdealer price P. We say that a distribution F first order stochastically dominates (FOSD) a distribution G if $F(x) \leq G(x)$ for all x, with strict inequality on a set of positive measure.

Proposition 2. *If Φ_1 and Φ_0 admit log-concave densities, the mappings $P \to B(\cdot|P)$ and $P \to A(\cdot|P)$ are strictly increasing with respect to FOSD.*

An increase in P shifts the bid distribution and yields uniformly better bid prices. A higher interdealer price raises realized profits of dealers and intensifies competition, which leads to more aggressive bidding. By symmetry, the ask distribution shifts downward, yielding uniformly lower ask prices, as a higher P reduces the surplus dealers can extract when selling to investors. This result is illustrated in figure 2.

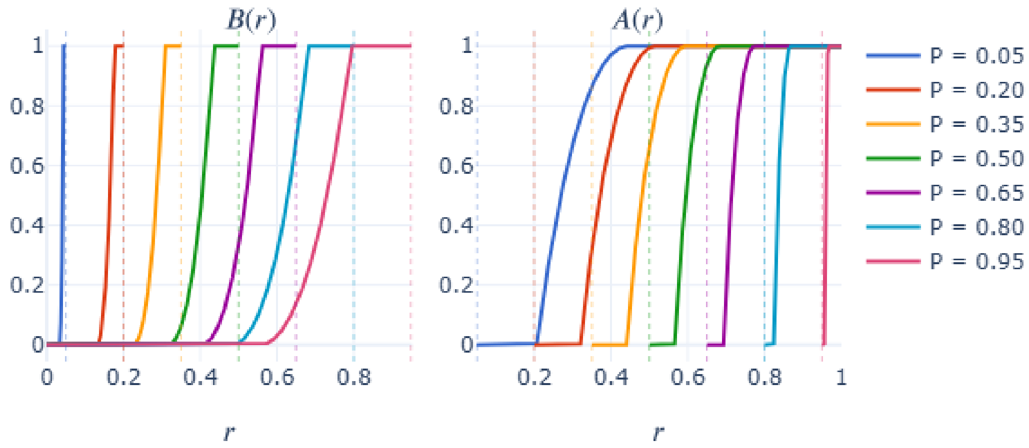

Figure 2: Bid and ask price distributions for different values of P in a static RFQ auction. $\Phi_1 = \Phi_0 = \text{Beta}(2,2)$ and $\Psi \sim \text{NB}(3.75, 0.9)|_{>0}$. An increase in P makes bid prices strictly more favorable by raising dealers' realized profits and intensifying competition. Conversely, it makes ask prices less favorable.

The auction environment exhibits a natural scale invariance. If Φ_0 and Φ_1, as well as P, are rescaled, the equilibrium bid and ask distributions B and A rescale accordingly. Moreover, there is no interaction between the bid and ask sides at this stage of the analysis. It is therefore relevant to study the normalized bid and ask sides of the auction. We define the normalized reservation price distributions

$$\tilde{\Phi}_1(z|P) \equiv \frac{\Phi_1(\underline{r} + z(P - \underline{r}))}{\Phi_1(P)} \quad \text{and} \quad \tilde{\Phi}_0(z|P) \equiv \frac{\Phi_0(P + z(\overline{r} - P)) - \Phi_0(P)}{1 - \Phi_0(P)}$$

for $z \in [0,1]$. These correspond to the distributions Φ_1 and Φ_0 restricted to $[\underline{r}, P]$ and $[P, \overline{r}]$, respectively, and rescaled to $[0,1]$. We also define $\tilde{B}(z|P)$ and $\tilde{A}(z|P)$ as the equilibrium bid and ask distributions associated with these normalized distributions. By scale invariance, they coincide with $B(r|P)$ and $A(r|P)$ under the corresponding change of variables. The next result shows how the dependence of $\tilde{\Phi}_1$ and $\tilde{\Phi}_0$ on P shapes $\tilde{B}(z|P)$ and $\tilde{A}(z|P)$.

Proposition 3. *Let Φ_1 and Φ_0 admit log-concave densities.*

If $\tilde{\Phi}_1(\cdot \mid P)$ is decreasing (increasing) in P with respect to FOSD, then $\tilde{B}(\cdot \mid P)$ is decreasing (increasing) in P with respect to FOSD. The monotonicity is strict whenever the FOSD shift in $\tilde{\Phi}_1$ is strict.

If $\tilde{\Phi}_0(\cdot \mid P)$ is decreasing (increasing) in P with respect to FOSD, then $\tilde{A}(\cdot \mid P)$ is increasing (decreasing) in P with respect to FOSD. The monotonicity is strict whenever the FOSD shift in $\tilde{\Phi}_0$ is strict.

The normalized distributions $\tilde{\Phi}_1$ and $\tilde{\Phi}_0$ place the distribution of investors on a common support, which isolates the composition effects from scale. They therefore govern the intensity of dealer competition. On the bid side, a shift of $\tilde{\Phi}_1$ toward higher valuations increases surplus and intensifies competition, leading to more aggressive bids. On the ask side, the effect is reversed: a shift of $\tilde{\Phi}_0$ toward higher valuations allows dealers to extract more surplus, resulting in worse ask prices. This result should be contrasted with proposition 2, which shows that $B(\cdot \mid P)$ shifts upward in the FOSD sense as P increases. This monotonicity reflects a scale effect: a higher interdealer price mechanically raises bid levels. By contrast, the normalized distributions isolate composition effects by fixing the support. As a result, shifts in $\tilde{B}(\cdot \mid P)$ capture how the *relative* aggressiveness of bid changes with P. In particular, even though bids increase in levels, they may become relatively less aggressive, depending on how $\tilde{\Phi}_1$ varies with P. Understanding the separation between scale and composition effects is crucial when we introduce the general equilibrium environment.

While the assumptions of proposition 3 are restrictive, lemma 16 in appendix A.2 gives equivalent conditions in terms of the rescaled (reverse) hazard rates

$$\tilde{m}_1(r) \equiv (r - \underline{r})\frac{\phi_1(r)}{\Phi_1(r)} \quad \text{and} \quad \tilde{m}_0(r) \equiv (\overline{r} - r)\frac{\phi_0(r)}{1 - \Phi_0(r)}.$$

Moreover, these assumptions are satisfied by several commonly used distributions, including the beta and truncated exponential distributions, as well as their reflected counterparts.

Lemma 3. *Let $\Phi = \Phi_1 = \Phi_0$.*

If $\Phi \sim Beta(a, b)$ with $a, b \geq 1$, then both $\tilde{\Phi}_1(\cdot \mid P)$ and $\tilde{\Phi}_0(\cdot \mid P)$ are decreasing in P with respect to FOSD.

If Φ is a truncated exponential distribution, then $\tilde{\Phi}_1(\cdot \mid P)$ is decreasing and $\tilde{\Phi}_0(\cdot \mid P)$ is increasing in P with respect to FOSD.

If Φ is a reflected truncated exponential distribution, then $\tilde{\Phi}_1(\cdot \mid P)$ is increasing and $\tilde{\Phi}_0(\cdot \mid P)$ is decreasing in P with respect to FOSD.

A simple benchmark is the uniform distribution. In this case, the normalized distributions do not depend on P, and composition effects vanish. Another useful example is the $\text{Beta}(a, 1)$ distribution. In this case, composition effects vanish on the bid side but remain present on the ask side. Figure 3 illustrates lemma 3 for a $\text{Beta}(2, 2)$ distribution. Figure 4 then shows how this translates into the rescaled price distributions, consistent with proposition 3. For a $\text{Beta}(2, 2)$ distribution, both the bid and ask price distributions B and A shift to the right in the FOSD sense as P increases (scale effect, figure 2), while the rescaled distributions $\tilde{B}$ and $\tilde{A}$ shift to the left (composition effect, figure 4).

We now study the effects of dealer competition. We compare dealer participation distributions using the likelihood ratio order, as in Jewitt (1991). We say that a distribution Ψ' is more competitive than Ψ, if $\Psi' \neq \Psi$ and the likelihood ratio $\frac{\psi'_n}{\psi_n}$ is weakly increasing in n whenever $\psi_n > 0$. We denote this ordering by $\Psi' \succ \Psi$.

Proposition 4. *If $\Psi' \succ \Psi$, then $B(\cdot|\Psi')$ strictly dominates $B(\cdot|\Psi)$ with respect to FOSD and $A(\cdot|\Psi)$ strictly dominates $A(\cdot|\Psi')$ with respect to FOSD.*

The intuition is straightforward: more competition yields more favorable prices for investors. On the bid side, this leads to higher bids, while on the ask side it results in lower asks. Importantly, the lower bound of the bid distribution, $\underline{r}_b$, and the upper bound of the ask distribution, $\bar{r}_a$, are independent of the competition structure, as they correspond to the maximizers of monopolistic profit on each side. In particular, $\underline{r}_b(\Psi') = \underline{r}_b(\Psi)$ and $\bar{r}_a(\Psi') = \bar{r}_a(\Psi)$. This property no longer holds once general equilibrium effects are introduced. By contrast, the other edges of the supports are pinned down by ν_1. Since $\nu'_1 < \nu_1$, it follows that $\bar{r}_b(\Psi') > \bar{r}_b(\Psi)$ and $\underline{r}_a(\Psi') < \underline{r}_a(\Psi)$. Hence, higher competition increases price dispersion, but only through the right edge of the bid distribution and the left edge of the ask distribution.

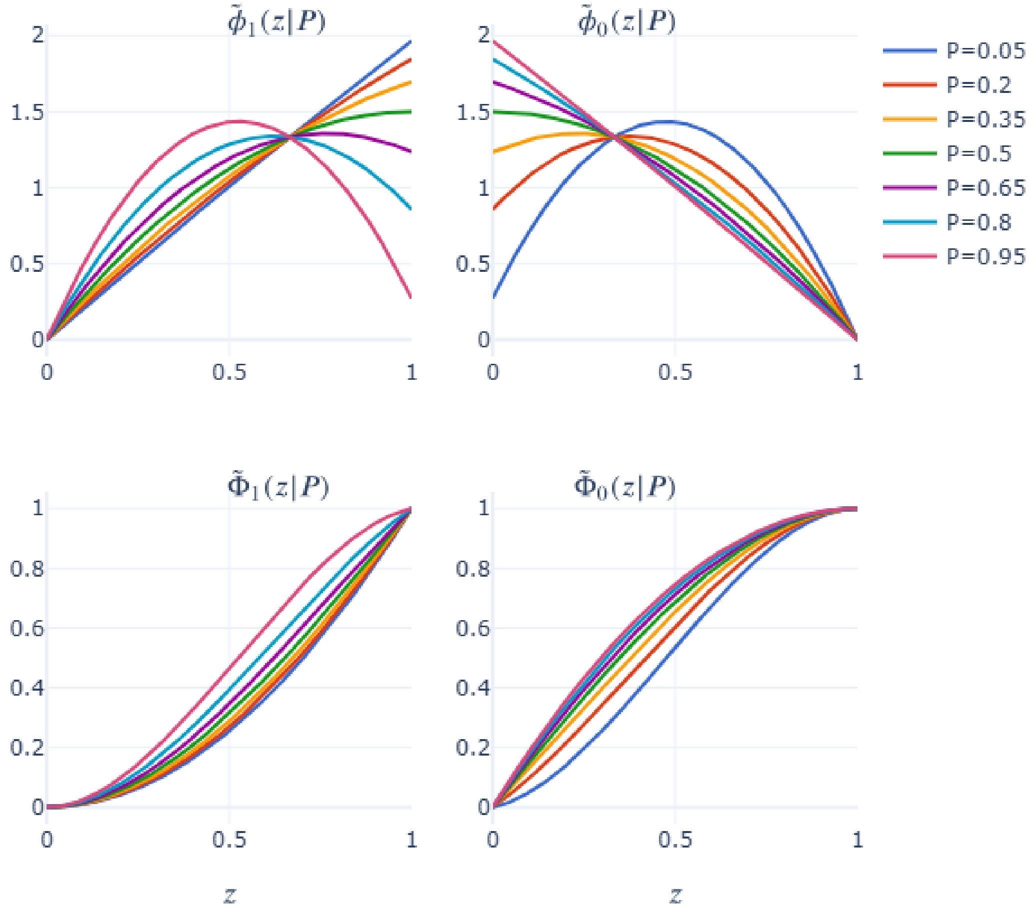

Figure 3: Rescaled $\tilde{\Phi}_1(\cdot \mid P)$ and $\tilde{\Phi}_0(\cdot \mid P)$ for different values of P when $\Phi_1 = \Phi_0 = \text{Beta}(2,2)$. The upper panels show the densities, while the lower panels display the corresponding CDFs. For Beta distributions, rescaled distributions shift downward in the FOSD order as P increases, as seen in the lower panels where the CDF for higher P lies above that for lower P.

3 Market model

In this section, we develop a dynamic search model of OTC markets in which the distributions of types introduced in the previous section are microfounded and endogenously determined in general equilibrium. Specifically, we consider a semi-centralized market populated by dealers and investors, in the spirit of DGP. Privately informed investors who wish to trade submit RFQs. Dealers compete by submitting price quotes and, upon execution, immediately offset their positions in an interdealer market.

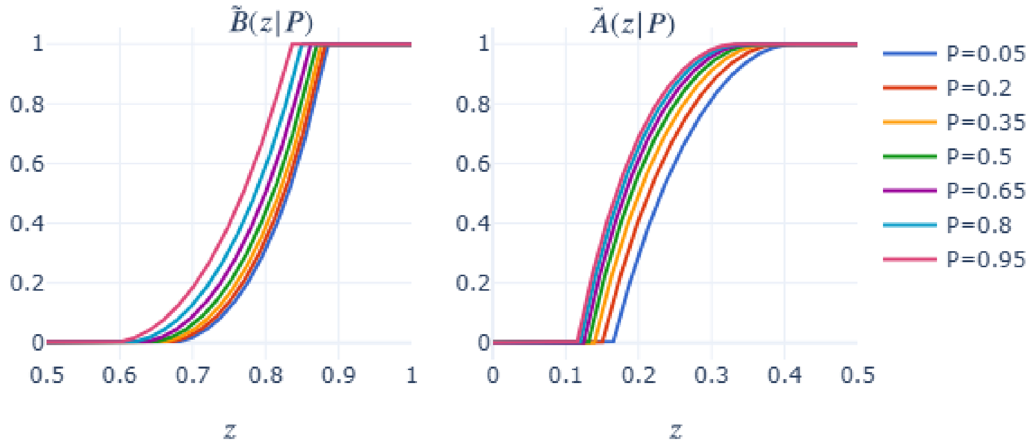

Figure 4: Rescaled price distributions $\tilde{B}(\cdot \mid P)$ and $\tilde{A}(\cdot \mid P)$ for different values of P when $\Phi_1 = \Phi_0 = \text{Beta}(2,2)$ and $\Psi \sim \text{NB}(3.75, 0.9)|_{>0}$. Both rescaled price distributions decrease in the FOSD order as P increases, implying that the relative quality of bids deteriorates while that of asks improves.

3.1 Asset and investors

Time is continuous and is indexed by $t \geq 0$. There is one tradable asset available in fixed supply $s \in (0,1)$. The economy is populated by a continuum of risk-neutral investors of total mass one who are infinitely lived and discount the time at a rate $r > 0$. Every investor holds either zero or one unit of the asset and enjoys a utility flow $q_t\delta_t$ from holding $q_t \in \{0,1\}$ unit of the assets at t. The utility flow δ_t is heterogeneous across investors and we refer to it as her *type*. It may capture differences in beliefs, preferences, or hedging needs.

We assume that δ_t can take values in a bounded interval $\mathcal{D} = [\underline{\delta}, \overline{\delta}]$ and evolves stochastically through time following a Markov jump process. For each investor, δ_t remains constant between the arrival times of a Poisson process with intensity $\gamma > 0$. At each arrival, δ_t is redrawn from a continuous distribution F on $\mathcal{D}$, independently across investors and over time. Without loss of generality, we assume that $\inf \operatorname{supp}(F) = \underline{\delta} > \sup \operatorname{supp}(F) = \overline{\delta}$.

Investors trade through RFQs with dealers. Contact between an investor and dealers occurs only at the arrival times of a Poisson process with intensity $\lambda > 0$. These processes are independent across investors and independent of the process governing investor types. As in Section 2, the number of participating dealers in each RFQ is an exogenous random variable $\Psi > 0$ taking values in $\mathbf{N}$ and is independent across RFQs. We define $g : [0, 1] \to [0, 1]$ the probability generating function of Ψ

$$g(q) \equiv \sum_{k=1}^{\infty} \psi_k q^k.$$

Each of the responding dealers draws a bid (respectively, ask) price independently from a distribution B_p (respectively, A_p), as characterized in Section 2. These distributions are determined endogenously in equilibrium and supported on $[\underline{r}, \overline{r}]$. The investor either selects the most favorable quote or rejects the trade. If she trades, she chooses the highest bid when selling and the lowest ask when buying. Standard arguments imply that the preferred bid and ask prices follow the distributions B_p^* and A_p^* characterized by

$$B_p^*(b) = g(B_p(b)) \quad \text{and} \quad 1 - A_p^*(a) = g(1 - A_p(a)).$$

The value function of investors $V : \{0, 1\} \times \mathcal{D} \to \mathbf{R}$ is defined by

$$V(q_t, \delta_t) \equiv \sup_q \mathbf{E}_t \left[\int_t^{\infty} e^{-r(u-t)} (q_u \delta_u du - P_u dq_u) \right],$$

and the reservation utility $R : \mathcal{D} \to \mathbf{R}$ by

$$R(\delta) \equiv V(1, \delta) - V(0, \delta),$$

which measures the utility gain from holding the asset. Standard results (e.g. Hugonnier et al. (2025)) imply that a unique solution exists, R is absolutely continuous, strictly increasing, and satisfies the HJB equation

$$\begin{aligned} rR(\delta) =& \delta + \gamma \int_{\underline{\delta}}^{\overline{\delta}} (R(x) - R(\delta)) dF(x) + \lambda \int_{\underline{r}}^{\overline{r}} (b - R(\delta))^+ dB_p^*(p) \\ & - \lambda \int_{\underline{r}}^{\overline{r}} (R(\delta) - a)^+ dA_p^*(a). \end{aligned} \tag{7}$$

The optimal strategy of the investor is to sell the asset at the preferred bid only if it is greater than the reservation value and to buy the asset at the preferred ask only if it is smaller than the reservation value. The reservation value $R(\delta)$ is the analogous to the reservation price r described in section 2.

3.2 Dealer intermediation

The economy is populated by a finite mass of dealers. In contrast to investors, dealers have access to a frictionless inter-dealer market in which they can trade the asset instantaneously at price P, determined in equilibrium to clear the market. Dealers act as intermediaries: upon trading with an investor, they immediately offset their position in the inter-dealer market and do not hold inventory. Their profits arise from trading with investors at prices different from P. When responding to an RFQ, dealers submit quotes according to the mechanism described in section 2.

Φ_0 and Φ_1 denote the measures over types δ and describe the allocation of the asset across investors, where $\Phi_0(\delta)$ (respectively, $\Phi_1(\delta)$) corresponds to the mass of investors holding zero (respectively, one) unit of the asset. We retain the notation of Section 2, although these objects are now defined in the space of types rather than prices. Moreover, they do not sum to one[4]. Under this notation, the dealer profit functions (4) and (5) in an RFQ auction can be written

$$\Pi_b^p(b|B_p) \equiv (P-b)\frac{\Phi_1(R^{-1}(b))}{\Phi_1(\bar{\delta})}h(B_p(b)) \tag{8}$$

and

$$\Pi_a^p(a|A_p) \equiv (a-P)\left(1-\frac{\Phi_0(R^{-1}(a))}{\Phi_0(\bar{\delta})}\right)h(1-A_p(a)). \tag{9}$$

To simplify the formulation of the problem, we assume without loss of generality in equilibrium that the interdealer price P and the optimal prices (a,b) lie in the range of the reservation value function $\mathcal{R} = \left[R(\underline{\delta}), R(\bar{\delta})\right]$. Indeed, if $P \notin \mathcal{R}$, the condition $b \leq P$ and $a \geq P$ imply that investors only trade in one direction and the market can not clear. Moreover, if $P \in \mathcal{R}$, choosing $b < \underline{R}$ is weakly dominated by choosing any b in the interval $[\underline{R}, P]$. The intuition is similar with a. We denote by $\delta^* \in \mathcal{D}$ the marginal type in the economy, defined as the unique solution of

$$R(\delta^*) = P.$$

[4]The equivalence is immediate: $\Phi_i^p(r) = \Phi_i(R^{-1}(r))/\Phi_i(\bar{\delta})$ for $i = 0, 1$.

Rather than working in price space as in section 2, we reformulate the problem in type space. In this representation, the bid and ask distributions B and A satisfy

$$B(\delta) \equiv B_p(R(\delta)) \quad \text{and} \quad A(\delta) \equiv A_p(R(\delta)).$$

Using this change of variable $r = R(\delta)$, the profit functions (8) and (9) can be expressed in type space as

$$\Pi_b(\delta|B) \equiv (R(\delta^*) - R(\delta)) \frac{\Phi_1(\delta)}{\Phi_1(\bar{\delta})} h(B(\delta))$$

and

$$\Pi_a(\delta|A) \equiv (R(\delta) - R(\delta^*)) \left(1 - \frac{\Phi_0(\delta)}{\Phi_0(\bar{\delta})}\right) h(1 - A(\delta)).$$

Following section 2, the equilibrium optimality conditions read

$$\begin{aligned} \text{supp}(B) &\subseteq \arg\max_{\delta \in [\underline{\delta}, \delta^*]} \Pi_b(\delta|B) \\ \text{supp}(A) &\subseteq \arg\max_{\delta \in [\delta^*, \bar{\delta}]} \Pi_a(\delta|A). \end{aligned} \tag{10}$$

When $\psi_1 = 1$, only one dealer responds to each RFQ. Investors meet a single dealer and trade bilaterally, without competition at the quoting stage. In this case, the RFQ mechanism reduces to the standard voice trading channel typically studied in the OTC literature. The model still features private information on the investor side and therefore differs from bargaining-based theories. The dealer effectively acts as a monopolist when offering a quote, but private information limits the extraction of the full surplus. From a modeling perspective, this case is equivalent to setting $g(q) = q$ and $h(q) = 1$, and the equilibrium structure is unchanged otherwise. More broadly, our framework is not restricted to RFQ trading and provides a general approach to studying private information in OTC markets.

3.3 Market clearing and stationarity

We focus on stationary equilibria in which the measures Φ_0 and Φ_1 are time-invariant. The market clearing condition requires that

$$\Phi_1(\bar{\delta}) = s. \tag{11}$$

By a law of large numbers argument, if the cross-sectional distribution of types is initially given by F, then it remains equal to F at all times. Hence,

$$\Phi_0(\delta) + \Phi_1(\delta) = F(\delta). \tag{12}$$

It follows that stationarity of Φ_1 is necessary and sufficient for stationarity of Φ_0.

To ensure that Φ_1 is stationary, we impose that for each $\delta \in \mathcal{D}$, the inflow into the set $\{q_t = 1, \delta_t \in d\delta\}$ equals the outflow from this set. The inflow arises from two sources: (i) investors with $(q_t = 0, \delta_t \in d\delta)$ who buy the asset, and (ii) investors with $(q_t = 1, \delta_t \notin d\delta)$ whose type changes to a value in $d\delta$. The outflow consists of investors with $(q_t = 1, \delta_t \in d\delta)$ who sell the asset or experience a change in type. Informally, this condition can be written as

$$(\gamma + \lambda(1 - B^*(\delta)))d\Phi_1(\delta) = \gamma\Phi_1(\bar{\delta})dF(\delta) + \lambda A^*(\delta)d\Phi_0(\delta), \tag{13}$$

where $B^*(\delta) \equiv g(B(\delta))$ and $1 - A^*(\delta) \equiv g(1 - A(\delta))$ denote the distributions of the highest bid and lowest ask in type space. $1 - B^*(\delta)$ measures the acceptance probability of a sell order for an investor of type δ and $A^*(\delta)$ measures the acceptance probability of a buy order. The left hand side of (13) is the outflow and the right hand side is the inflow.

To simplify notations, we define $\tilde{g} : [0,1] \to [0,1]$ by

$$\tilde{g}(q) \equiv 1 - g(1 - q),$$

and (13) can be written

$$(\gamma + \lambda\tilde{g}(1 - B(\delta)))d\Phi_1(\delta) = \gamma\Phi_1(\bar{\delta})dF(\delta) + \lambda\tilde{g}(A(\delta))d\Phi_0(\delta). \tag{14}$$

Formally, this stationarity condition can be expressed as

$$\int_G (\gamma + \lambda \tilde{g}(1 - B(\delta))) d\Phi_1(\delta) = \int_G \gamma \Phi_1(\bar{\delta}) dF(\delta) + \int_G \lambda \tilde{g}(A(\delta)) d\Phi_0(\delta) \tag{15}$$

for any Borel set $G \subseteq \mathcal{D}$. The formal foundations for the arguments in this section are provided by Duffie and Sun (2012).

Remark 1. We assume that at least one dealer responds to each RFQ. This is without loss of generality. To see this, consider an alternative specification in which no dealer responds with probability $\psi_0 > 0$. In that case, conditional on at least one response, the number of participating dealers follows the distribution $\{\psi'_k\}_{k \geq 1}$ defined by $\psi'_k = \psi_k / (1 - \psi_0)$ and RFQs with at least one response arrive at rate $\lambda' = (1 - \psi_0)\lambda$. Hence, the model is equivalent to one in which every RFQ receives at least one response, with arrival rate λ' and distribution $\{\psi'_k\}_{k \geq 1}$. In this sense, λ should be interpreted as the effective rate at which investors receive at least one quote.

Remark 2. Allowing for an arbitrary distribution Ψ enables the model to capture situations in which investors submit RFQs across multiple trading platforms. For instance, suppose that on each platform the number of responding dealers follows a binomial distribution and that investors submit RFQs to two platforms simultaneously. Then Ψ corresponds to the distribution of the sum of two independent binomial random variables when both platforms provide quotes simultaneously.

Remark 3. An OLG version of the model with bequests can be considered. Instead of investors changing types over time, agents die at rate γ and are replaced by children who inherit their asset holdings. Newborns draw their utility flow from the distribution $F(\delta)$. The only modification relative to the baseline model is the HJB equation (7), which becomes

$$(r + \gamma) R(\delta) = \delta + \lambda \int_{\underline{p}}^{\overline{p}} (p - R(\delta))^+ d\tilde{B}^*(p) - \lambda \int_{\underline{p}}^{\overline{p}} (R(\delta) - p)^+ d\tilde{A}^*(p).$$

The rest of the analysis is unchanged. While less common in the literature, this formulation has the advantage that the equilibrium interdealer price satisfies $P = \frac{\delta^*}{r+\gamma}$, which provides a convenient mapping between the elasticity of the marginal type and the price elasticity.

Remark 4. In DGP and related bargaining-based OTC models, prices are invariant to the cross-sectional distribution of asset holdings, as long as $\Phi_{0t}(\delta) + \Phi_{1t}(\delta) =$

$F(\delta)$. This property no longer holds in our framework. Here, the distribution of asset holdings directly enters in the pricing mechanism.

4 Market equilibrium

4.1 Definition

We now define a stationary equilibrium as a collection of a reservation value function, bid and ask price distributions, measures of asset allocations, and an interdealer price such that investors and dealers behave optimally and the market clears.

Definition 2. *A stationary equilibrium consists of a reservation value function R, a pair of pricing strategies (B, A), a pair of cumulative measures (Φ_0, Φ_1), and an interdealer price $P = R(\delta^*)$ such that*

1. *$R(\delta)$ satisfies (7) given B and A,*
2. *A and B satisfy (10) given R, δ^*, Φ_0 and Φ_1,*
3. *Φ_0 and Φ_1 satisfy (11), (12) and (15) given B and A.*

In the next section, we show that each component of the definition can be solved in closed-form, taking the other equilibrium objects as given. The main difficulty lies in jointly verifying all three conditions, which requires solving an infinite dimensional equation.

4.2 Partial characterization

We first solve for the reservation value and the stationary allocation, corresponding to the first and third conditions of definition 2, taking B and A as given. The HJB equation (7) can be rewritten as

$$\begin{aligned} rR(\delta) =&\delta + \gamma \int_{\underline{\delta}}^{\overline{\delta}} (R(x) - R(\delta))dF(x) + \lambda \int_{\delta}^{\overline{\delta}} (R(x) - R(\delta))dB^*(x) \\ &- \lambda \int_{\underline{\delta}}^{\delta} (R(\delta) - R(x))dA^*(x), \end{aligned} \tag{16}$$

which is solved in closed-form up to the constant $R(\delta^*)$.

Lemma 4. *The solution to* (16) *is*

$$R(\delta) = R(\delta^*) + \int_{\delta^*}^{\delta} \frac{1}{r + \gamma + \lambda(\tilde{g}(1 - B(x)) + \tilde{g}(A(x)))} dx. \tag{17}$$

Proof. The proof is similar to the proof of Lemma 10.1 in Hugonnier et al. (2025), replacing B and A with B^* and A^*. ■

The expression in (17) provides an interpretable characterization of the reservation value as the accumulation of marginal gains from holding the asset across types. The parameter γ flattens the slope of R, reflecting the fact that types matter less when preference shocks occur more frequently. The terms $\lambda\tilde{g}(1 - B(\delta))$ and $\lambda\tilde{g}(A(\delta))$ capture the effective rate at which investors of type δ can trade. For a given $P = R(\delta^*)$, better selling conditions increase the reservation value by raising the continuation value $V(1, \cdot)$ of holding the asset. Conversely, better buying conditions reduce the reservation value by increasing the value $V(0, \cdot)$ of not holding the asset. Overall, improved trading conditions compress differences in continuation values across types, which flattens the reservation value function.

Lemma 5. *The solution to* (15) *subject to* (12) *is*

$$\Phi_1(\delta) = \Phi_1(\delta|m) \equiv \int_{\underline{\delta}}^{\delta} \frac{\gamma m + \lambda\tilde{g}(A(x))}{\gamma + \lambda(\tilde{g}(1 - B(x)) + \tilde{g}(A(x)))} dF(x) \tag{18}$$

where $m \in [0, 1]$ *is a solution to* $\Phi_1(\bar{\delta}|m) = m$. *The expression for* Φ_0 *is*

$$\Phi_0(\delta) = \int_{\underline{\delta}}^{\delta} \frac{\gamma(1 - m) + \lambda\tilde{g}(1 - B(x))}{\gamma + \lambda(\tilde{g}(1 - B(x)) + \tilde{g}(A(x)))} dF(x)$$

If $(B, A) \neq (\mathbf{1}_{\delta \geq \underline{\delta}}, \mathbf{1}_{\delta \geq \bar{\delta}})$, *m is unique. If* $m \in (0, 1)$, Φ_1 *and* Φ_0 *are equivalent to* F.

Proof. The expression follows by substituting (12) and (11) into (14) and solving the resulting equation for $d\Phi_1(\delta)$. See Lemma A.14 of Hugonnier et al. (2025) for a formal argument. m uniquely exists since $m - \Phi_1(\bar{\delta}|m)$ is continuous in m, $0 - \Phi_1(\bar{\delta}|0) \leq 0$, $1 - \Phi_1(\bar{\delta}|1) \geq 0$ and $\frac{d}{dm}\left(m - \Phi_1(\bar{\delta}|m)\right) > 0$, as long as $B \neq \mathbf{1}_{\delta \geq \underline{\delta}}$ and $A \neq \mathbf{1}_{\delta \geq \bar{\delta}}$. The last assertion follows from the positivity of the integrand. ■

Lemma 5 characterizes the endogenous measures of asset allocation across types. For a given mass m of holders, the integrand in (18) gives the fraction of type x investors who hold the asset in the stationary allocation. Better selling conditions reduce Φ_1

at low valuations, since low-type holders can unwind their position more easily and therefore spend less time holding the asset. Symmetrically, better buying conditions increase Φ_1 at high valuations, since high-type investors can acquire the asset more easily and therefore spend more time in the holding state. As a result, trading conditions shape not only prices but also the composition of asset holders across types. In equilibrium, market clearing condition requires $m = s$.

The second equilibrium condition of definition 2 requires that A and B satisfy (10) given R, δ^*, Φ_0, and Φ_1. This problem has been solved in section 2. We summarize the relevant results in lemma 6. To express the solution, we define

$$\hat{\Pi}_b(\delta) \equiv (R(\delta^*) - R(\delta))\frac{\Phi_1(\delta)}{\Phi_1(\bar{\delta})} \quad , \quad \hat{\Pi}_a(\delta) \equiv (R(\delta) - R(\delta^*))\left(1 - \frac{\Phi_0(\delta)}{\Phi_0(\bar{\delta})}\right), \tag{19}$$

as well as

$$\hat{\Pi}_b^*(\delta) \equiv \sup_{x\in\left[\delta,\bar{\delta}\right]} \hat{\Pi}_b(x) \quad \text{and} \quad \hat{\Pi}_a^*(\delta) \equiv \sup_{x\in[\underline{\delta},\delta]} \hat{\Pi}_a(x).$$

Lemma 6. *The bid and ask price distributions B and A solving (10) satisfy*

- *If* $\psi_1 = 0$, $B(\delta) = A(\delta) = \mathbf{1}_{\delta\geq\delta^*}$.
- *If* $\delta^* = \underline{\delta}$, *then* $B(x) = \mathbf{1}_{\delta\geq\delta^*}$. *If* $\delta^* = \bar{\delta}$, *then* $A(x) = \mathbf{1}_{\delta\geq\delta^*}$.
- *If* $\psi_1 \in (0,1)$ *and* $\delta^* > \underline{\delta}$, $B(\delta) = \tilde{\mathbf{h}}\left(\hat{\Pi}_b^*(\underline{\delta}), \hat{\Pi}_b^*(\delta)\right)$.
- *If* $\psi_1 \in (0,1)$ *and* $\delta^* < \bar{\delta}$, $A(\delta) = 1 - \tilde{\mathbf{h}}\left(\hat{\Pi}_a^*(\bar{\delta}), \hat{\Pi}_a^*(\delta)\right)$.
- *If* $\psi_1 = 1$, *supp*$(B) \subseteq \operatorname{argmax}_{\delta\in[\underline{\delta},\delta^*]} \hat{\Pi}_b(\delta)$ *and supp*$(A) \subseteq \operatorname{argmax}_{\delta\in\left[\delta^*,\bar{\delta}\right]} \hat{\Pi}_a(\delta)$.

Remark 5. All the analysis developed so far extends to distributions F with atoms. When the marginal type δ^* lies in the interior of $\mathcal{D}$, the characterization is unchanged, as δ^* adjusts to absorb any mass points. If δ^* lies at the boundary of $\mathcal{D}$, market clearing requires randomization by agents at the margin, i.e. they only accept trades with some endogenous probability.

4.3 Existence

In this section, we establish the existence of an equilibrium. When $\psi_1 = 0$, the model reduces to DGP with full bargaining power for investors, and the existence and uniqueness have already been established. We therefore restrict our attention to $\psi_1 > 0$. The continuity of F is crucial for the arguments in this section.

Theorem 1. *A stationary equilibrium exists when* $\psi_1 > 0$.

The details of the proof are in Appendix B. The main steps are outlined below. We construct a price-adjustment operator on the joint space of price distributions and marginal types, and we establish the existence of a fixed point. The operator jointly determines price distributions and the marginal type, where the marginal type is adjusted in the direction of excess supply or demand. This approach is in the spirit of classical general equilibrium theory, but is not standard in the OTC literature.

Specifically, we define the bid-price operator $\mathcal{B}$, which takes (B, δ^*) as input and returns the optimal bid distribution of dealers when R and Φ_1 satisfy (17) and (18). If $\psi_1 = 1$, $\mathcal{B}$ is a correspondence and we use Berge maximum theorem to show that it is upper hemicontinuous with non-empty and convex values. If $\psi_1 < 1$, the optimal bid distribution is unique and $\mathcal{B}$ is a continuous operator. We proceed similarly for the ask price operator $\mathcal{A}$. We also define the marginal type update operator

$$\Delta^*(B, A, \delta^*) \equiv \delta^* + (\mathcal{S}(B, A) - s)^+ (\bar{\delta} - \delta^*) - (s - \mathcal{S}(B, A))^+ (\delta^* - \underline{\delta}),$$

where $\mathcal{S}(B, A)$ denotes the supply implied by (B, A).

We then construct the equilibrium operator (extending $\mathcal{B}$ and $\mathcal{A}$ to singleton valued correspondences when $\psi_1 < 1$)

$$\mathcal{E}(B, A, \delta^*) \equiv \mathcal{B}(B, \delta^*) \times \mathcal{A}(A, \delta^*) \times \{\Delta^*(B, A, \delta^*)\},$$

and apply Kakutani fixed-point theorem to establish the existence of a fixed point. We finally show that any fixed point of $\mathcal{E}$ defines an equilibrium.

5 Equilibrium characterization

In this section, we derive necessary conditions that any equilibrium must satisfy, establish uniqueness, and provide a numerical procedure for its computation. All results in this section are stated for $\psi_1 > 0$. For most of the analysis, we assume that F admits a density f that is log-concave.

5.1 Necessary and sufficient conditions

For now, we assume that F is continuously differentiable with density $f = F' > 0$ continuous. This already imposes useful restrictions on the equilibrium.

Proposition 5. *In equilibrium:*

1. *The marginal type δ^* lies in $(\underline{\delta}, \overline{\delta})$,*
2. $\text{supp}(B) \subset (\underline{\delta}, \delta^*)$ *and* $\text{supp}(A) \subset (\delta^*, \overline{\delta})$,
3. *B and A are continuous distributions.*

Proof. Suppose that $\delta^* = \underline{\delta}$. Then $B(\delta) = \mathbf{1}_{\{\delta \geq \underline{\delta}\}}$ and $A \neq \mathbf{1}_{\{\delta \geq \overline{\delta}\}}$. By lemma 5, the implied supply is equal to $1 > s$, which yields a contradiction. The argument for $\delta^* = \overline{\delta}$ is similar. The remaining statements follow from proposition 1 and proposition 10.6 in Hugonnier et al. (2025). ■

The interpretation of the first two statements is straightforward. In a stationary equilibrium, market clearing with a supply $s \in (0, 1)$ requires a strictly positive mass of both buyers and sellers, which rules out $\delta^* = \underline{\delta}$ or $\delta^* = \overline{\delta}$. The second statement follows from the fact that dealers have a strictly positive probability of being the sole respondent to an RFQ and can therefore extract a strictly positive surplus. When $\psi_1 \in (0, 1)$, the continuity of B and A follows directly from the static effects of competition, as established in lemma 1. When $\psi_1 = 1$ however, the intuition is less immediate, since the corresponding static pricing problem does not rule out atoms. To see why atoms cannot arise in equilibrium, consider an hypothetical equilibrium where dealers offer a single bid price $\hat{\delta}$. Then, upon meeting a dealer, all asset holders with $\delta < \hat{\delta}$ sell, while those with $\delta > \hat{\delta}$ do not. As a result, (18) implies a jump in the slope of Φ_1 at $\hat{\delta}$, leading to a local excess mass of asset holders just above $\hat{\delta}$. Dealers can then profitably deviate by bidding slightly above $\hat{\delta}$, to capture this mass while sacrificing only a negligible amount of surplus. This contradicts the optimality of the pricing strategy.

An implication of proposition 5 is the existence of thresholds $\underline{\delta}_b$, $\overline{\delta}_b$, $\underline{\delta}_a$, $\overline{\delta}_a$ such that

$$\underline{\delta} < \underline{\delta}_b < \overline{\delta}_b < \delta^* < \underline{\delta}_a < \overline{\delta}_a < \overline{\delta},$$

with $\text{supp}(B) \subseteq [\underline{\delta}_b, \overline{\delta}_b]$ and $\text{supp}(A) \subseteq [\underline{\delta}_a, \overline{\delta}_a]$. As in the static problem when $\psi_1 \in (0, 1)$, the bid and ask sides can be partitioned into three regions. Asset owners with $\delta < \underline{\delta}_b$ and non-owners with $\delta > \overline{\delta}_a$ have the largest gains from trade and are willing to trade immediately. Upon meeting a dealer, they trade with probability one. Asset owners with $\delta \in (\underline{\delta}_b, \overline{\delta}_b)$ and non-owners with $\delta \in (\underline{\delta}_a, \overline{\delta}_a)$ have intermediate gains from trade and are willing to forego immediacy in order to obtain better terms in future meetings. Upon meeting a dealer, they trade with a probability strictly between 0 and 1, which depends on their type. Finally, asset owners with $\delta > \overline{\delta}_b$ and non-owners with $\delta < \underline{\delta}_a$ have low gains from trade and do not trade. Both dealer competition and private information mitigate the market power of dealers within each match. These three regions are illustrated in figure 5.

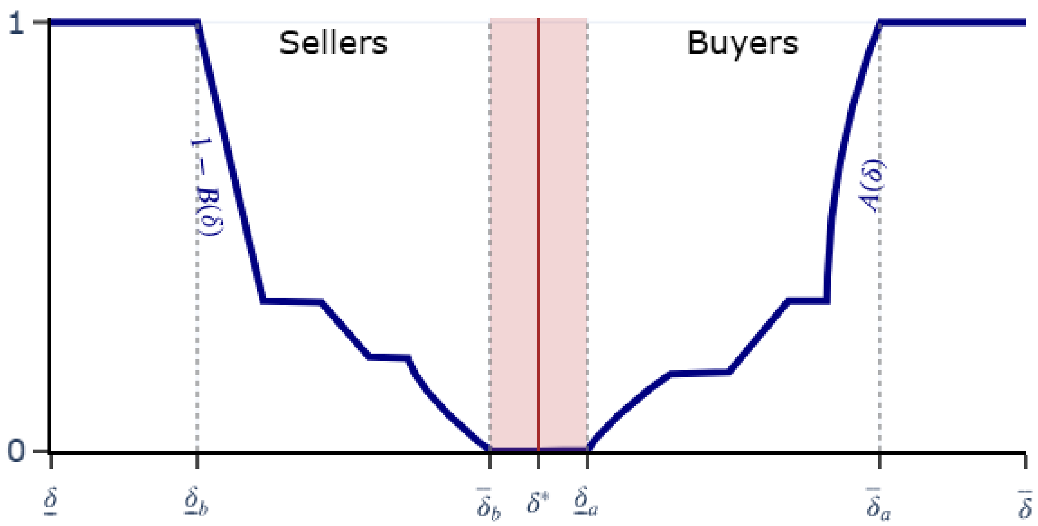

Figure 5: Illustration of the equilibrium bid and ask price distributions. Both sides of the market are partitioned into three regions. The no-trade region (shaded in red) corresponds to types with low gains from trade, for which information frictions prevent transactions. The intermediate region features types who trade with a strictly positive but less than one probability, reflecting the option value of waiting for better terms. Finally, in the extreme regions, investors trade immediately upon meeting a dealer. These specific distributions also feature ironing, which corresponds to the flat regions in $(\underline{\delta}_b, \overline{\delta}_a)$ and $(\underline{\delta}_b, \overline{\delta}_a)$.

Proposition 5 does not rule out the presence of gaps in the supports of the distributions, as shown in figure 5. In the mechanism design literature, this phenomenon is referred to as ironing. To rule out ironing in equilibrium and simplify the characterization, we assume that f is log-concave.

Lemma 7. *If the density f is log-concave, B and A have connected supports in equilibrium and there is no ironing.*

The proof is in appendix C.1. In contrast to lemma 2, the log-concavity assumption is imposed on f, rather than on Φ_0 and Φ_1, which govern adverse selection. In particular, Φ_0 and Φ_1 need not be log-concave and, in general, are not. For the remainder of the paper, we assume that f is log-concave. Lemma 7 implies that $\text{supp}(B) = [\underline{\delta}_b, \overline{\delta}_b]$ and $\text{supp}(A) = [\underline{\delta}_a, \overline{\delta}_a]$. We denote by π_b^* and π_a^* the maximum bid and ask profits of dealers.

$$\pi_b^* \equiv \max_{\delta \in \mathcal{D}} \Pi_b(\delta) > 0$$
$$\pi_a^* \equiv \max_{\delta \in \mathcal{D}} \Pi_a(\delta) > 0.$$

Importantly, π_b^* and π_a^* pin down $\underline{\delta}_b$ and $\overline{\delta}_a$.

Lemma 8. *In equilibrium*

$$(r+\gamma+\lambda)\frac{\gamma+\lambda}{\gamma}\frac{\pi_b^*}{\nu_1} = \frac{F(\underline{\delta}_b)}{f(\underline{\delta}_b)}F(\underline{\delta}_b) \tag{20}$$
$$(r+\gamma+\lambda)\frac{\gamma+\lambda}{\gamma}\frac{\pi_a^*}{\nu_1} = \frac{1-F(\overline{\delta}_a)}{f(\overline{\delta}_a)}(1-F(\overline{\delta}_a)).$$

In particular, there exists a unique pair $(\underline{\delta}_b, \overline{\delta}_a) \in (\underline{\delta}, \overline{\delta})^2$ *that solves the above equation given any* $\pi_b^* \in (0, \overline{\pi}_b)$ *and* $\pi_a^* \in (0, \overline{\pi}_a)$*, where* $\overline{\pi}_b \equiv \frac{\nu_1}{f(\overline{\delta})}\frac{\gamma}{(\gamma+\lambda)(r+\gamma+\lambda)}$ *and* $\overline{\pi}_a \equiv \frac{\nu_1}{f(\underline{\delta})}\frac{\gamma}{(\gamma+\lambda)(r+\gamma+\lambda)}$.

The proof is provided in appendix C and relies on the fact that the lower bound of the support of B coincides with the maximizer of $\hat{\Pi}_b$. We denote by $\underline{\delta}_b(\pi_b^*)$ and $\underline{\delta}_a(\pi_a^*)$ the unique solutions to (20) given π_b^* and π_a^*. The next lemma provides a functional characterization of the equilibrium bid and ask distributions. Combining the optimality conditions of dealers, with the expressions for the reservation value and the stationary allocation, we obtain a system of integral equations that must be satisfied by B and A on their respective supports. These equations express the fact that dealers are indifferent across all bids or asks in the support.

Lemma 9. *In equilibrium, for all* $\delta \in [\underline{\delta}_b, \overline{\delta}_b]$

$$\pi_b^* = \left(\frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\delta}\frac{1}{r+\gamma+\lambda\tilde{g}(1-B(x))}dx\right) \tag{21}$$
$$\cdot\left(\frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \int_{\underline{\delta}_b}^{\delta}\frac{\gamma}{\gamma+\lambda\tilde{g}(1-B(x))}f(x)dx\right)h(B(\delta)),$$

and for all $\delta \in [\underline{\delta}_a, \overline{\delta}_a]$

$$\pi_a^* = \left(\frac{\pi_a^*}{\nu_1} \frac{\gamma + \lambda}{\gamma} \frac{1}{1 - F(\overline{\delta}_a)} - \int_\delta^{\overline{\delta}_a} \frac{1}{r + \gamma + \lambda \tilde{g}(A(x))} dx \right) \cdot \left(\frac{\gamma}{\gamma + \lambda} (1 - F(\overline{\delta}_a)) + \int_\delta^{\overline{\delta}_a} \frac{\gamma}{\gamma + \lambda \tilde{g}(A(x))} f(x) dx \right) h(1 - A(\delta)). \tag{22}$$

Another key result is that the spreads $\delta^* - \overline{\delta}_b$ and $\underline{\delta}_a - \delta^*$ are explicit functions of (π_b^*, B) and (π_a^*, A).

Lemma 10. *In equilibrium*

$$\delta^* = \overline{\delta}_b + \frac{r + \gamma}{\int_{\underline{\delta}}^{\overline{\delta}_b} \frac{\gamma f(x)}{\gamma + \lambda \tilde{g}(1 - B(x))} dx} \pi_b^* = \underline{\delta}_a - \frac{r + \gamma}{\int_{\underline{\delta}_a}^{\overline{\delta}} \frac{\gamma f(x)}{\gamma + \lambda \tilde{g}(A(x))} dx} \pi_a^*. \tag{23}$$

To construct an equilibrium, one first fixes $\pi_b^* \in (0, \overline{\pi}_b)$ and $\pi_a^* \in (0, \overline{\pi}_a)$. The corresponding $\underline{\delta}_b$ and $\overline{\delta}_a$ are then determined using Lemma 8. The next step is to find increasing functions B and A, together with $\overline{\delta}_b$ and $\underline{\delta}_a$, that solve the functional equations in Lemma 9 and satisfy the boundary conditions $B(\underline{\delta}_b) = 0$, $B(\overline{\delta}_b) = 1$, $A(\underline{\delta}_a) = 0$, and $A(\overline{\delta}_a) = 1$. Finally, one verifies that the interdealer prices implied by the bid and ask sides, computed using Lemma 10, coincide. If these conditions are satisfied, Proposition 6 ensures that they characterize an equilibrium if the market clears.

Proposition 6. *Let B and A be continuous distributions with supp$(B) = [\underline{\delta}_b, \overline{\delta}_b]$ and supp$(A) = [\underline{\delta}_a, \overline{\delta}_a]$. B and A form an equilibrium if and only if there exists (π_b^*, π_a^*) such that* (20), (23), (21), (22) *and*

$$s = \int_{\underline{\delta}}^{\overline{\delta}} \frac{\gamma s + \lambda \tilde{g}(A(x))}{\gamma + \lambda(\tilde{g}(1 - B(x)) + \tilde{g}(A(x)))} f(x) dx \tag{24}$$

are verified.

The details of the proof are in appendix C. The existence result in Theorem 1 ensures that a solution to the problem described in proposition 6 exists. However, solving this problem is non-trivial, and uniqueness is not immediate. In the next subsection, we establish several properties of the functional equations (21) and (22) that greatly simplify the search for an equilibrium and allow us to prove uniqueness.

5.2 Uniqueness

This subsection establishes uniqueness of equilibrium by reducing the infinite dimensional fixed-point problem to a two dimensional problem. While equilibrium is initially defined over distributions B and A, we show that, for any given (π_b^*, π_a^*), the functional equations (21) and (22) admit a unique solution. This reduction implies that equilibrium outcomes are fully characterized by (π_b^*, π_a^*). We then prove that the mappings from these parameters to aggregate supply and marginal type are injective, which establishes uniqueness of equilibrium. We impose the additional assumption that $\psi_2 > 0$ whenever $\psi_1 < 1$, which guarantees that h^{-1} is Lipschitz. Economically, this condition rules out local monopoly situations and is standard in related environments (see e.g. Garrett et al. (2019)).

Lemma 11. *For all $\pi_b^* \in (0, \overline{\pi}_b)$ and $\pi_a^* \in (0, \overline{\pi}_a)$, there exists a unique quadruple $(B, \overline{\delta}_b, A, \underline{\delta}_a)$ such that:*

1. $\overline{\delta}_b \in \left(\underline{\delta}_b(\pi_b^*), \overline{\delta}\right]$ *and* $\underline{\delta}_a \in \left[\underline{\delta}, \overline{\delta}_a(\pi_a^*)\right)$,
2. $B : \left[\underline{\delta}_b(\pi_b^*), \overline{\delta}_b\right] \to [0, 1]$ *and* $A : \left[\underline{\delta}_a, \overline{\delta}_a(\pi_a^*)\right] \to [0, 1]$ *are continuous and strictly increasing,*
3. *B and A solve* (21) *and* (22) *on their respective domains,*
4. $B(\underline{\delta}_b(\pi_b^*)) = 0$ *and* $A(\overline{\delta}_a(\pi_a^*)) = 1$,
5. *either* $B(\overline{\delta}_b) = 1$ *or* $\overline{\delta}_b = \overline{\delta}$, *and either* $A(\underline{\delta}_a) = 0$ *or* $\underline{\delta}_a = \underline{\delta}$.

The proof is provided in appendix C.2, and we outline the main steps here. When $\psi_1 = 1$, we transform the functional equation into a Riccati equation by introducing an auxiliary function. Uniqueness of the solution then follows from standard results, and we subsequently show that the solution is strictly increasing. When $\psi_1 < 1$, we rewrite the functional equation as a fixed-point problem. We then adapt the Picard–Lindelöf argument for uniqueness of solutions to ordinary differential equations to this setting. Finally, we show that any solution must be strictly increasing.

This result allows us to parametrize equilibrium objects by (π_b^*, π_a^*). In particular, we define the mappings $B(\cdot|\pi_b^*)$, $\overline{\delta}_b(\pi_b^*)$, $A(\cdot|\pi_a^*)$, and $\underline{\delta}_a(\pi_a^*)$ as the unique solutions to (21) and (22). We extend $B(\cdot|\pi_b^*)$ and $A(\cdot|\pi_a^*)$ to the entire domain $\mathcal{D}$ by assigning values 0 and 1 outside their supports. These mappings in turn define the implied interdealer prices $\delta_b^*(\pi_b^*)$ and $\delta_a^*(\pi_a^*)$ from the bid and ask sides, given by (23). The supply implied by (π_b^*, π_b^*) is formally defined as

$$s(\pi_b^*, \pi_a^*) \equiv \frac{\int_{\underline{\delta}}^{\overline{\delta}} \frac{\lambda \tilde{g}(A(x|\pi_a^*))}{\gamma+\lambda(\tilde{g}(1-B(x|\pi_b^*))+\tilde{g}(A(x|\pi_a^*)))} f(x)dx}{1-\gamma \int_{\underline{\delta}}^{\overline{\delta}} \frac{1}{\gamma+\lambda(\tilde{g}(1-B(x|\pi_b^*))+\tilde{g}(A(x|\pi_a^*)))} f(x)dx}, \tag{25}$$

which solves (24) for s. The next result shows that all these mappings are strictly monotone in (π_b^*, π_a^*).

Lemma 12. *Let $\pi_b^{(2)} > \pi_b^{(1)}$ and $\pi_a^{(2)} > \pi_a^{(1)}$. Then:*

1. *$B(\delta|\pi_b^{(2)})$ strictly dominates $B(\delta|\pi_b^{(1)})$ with respect to FOSD.*
   $A(\delta|\pi_a^{(1)})$ strictly dominates $A(\delta|\pi_a^{(2)})$ with respect to FOSD.

2. $$\delta_b^*(\pi_b^{(2)}) > \delta_b^*(\pi_b^{(1)}) \quad \textit{and} \quad \delta_a^*(\pi_a^{(2)}) < \delta_a^*(\pi_a^{(1)}).$$

3. *For all (π_b^*, π_a^*),*
   $$s(\pi_b^{(2)}, \pi_a^*) < s(\pi_b^{(1)}, \pi_a^*) \quad \textit{and} \quad s(\pi_b^*, \pi_a^{(2)}) > s(\pi_b^*, \pi_a^{(1)}).$$

The details of the proof are in appendix C.2. The bid and ask distributions vary monotonically in the sense of FOSD with respect to dealer expected profits π_b^* and π_a^*. Specifically, when the expected bid profit increases, investors sell the asset more frequently and the marginal type increases. These forces are standard and intuitive in the static environment of section 2 and correspond to the scale effects. In our dynamic general equilibrium setting, however, additional effects arise. Greater selling participation shrinks dealers' realized gains from trade $R(\delta^*) - R(\delta)$. Moreover, it shifts the distribution of asset holdings Φ_1 toward the margin, further reducing the expected surplus. These dynamic forces oppose the static ones. The proposition shows that the static forces dominate in equilibrium: higher expected profits are associated with greater investor participation and a higher marginal type.

Practically, lemma 12 serves two purposes. First, it simplifies the equilibrium search by indicating how π_b^* and π_a^* must be adjusted to satisfy $\delta_b^*(\pi_b^*) = \delta_a^*(\pi_a^*)$ and $s = s(\pi_b^*, \pi_a^*)$. Second, it provides the key monotonicity properties needed to establish uniqueness of equilibrium.

Theorem 2. *There exists a unique stationary equilibrium.*

Proof. Theorem 1 ensures existence of at least one equilibrium. Let B and A denote the price distributions of an equilibrium. By Proposition 6, there exist π_b^* and π_a^* such that

$$\delta_b^*(\pi_b^*) = \delta_a^*(\pi_a^*) \quad \text{and} \quad s = s(\pi_b^*, \pi_a^*).$$

Lemma 11 implies that B and A are uniquely determined by (π_b^*, π_a^*). Suppose, for contradiction, that there exists another equilibrium associated with $(\pi_b^{*'}, \pi_a^{*'}) \neq (\pi_b^*, \pi_a^*)$. Then either $\pi_b^{*'} \neq \pi_b^*$ or $\pi_a^{*'} \neq \pi_a^*$. Without loss of generality, assume $\pi_b^{*'} > \pi_b^*$. By lemma 12, this implies $\pi_a^{*'} < \pi_a^*$ in order to satisfy

$$\delta_b^*(\pi_b^{*'}) = \delta_a^*(\pi_a^{*'}).$$

It then follows again from lemma 12 that

$$s(\pi_b^{*'}, \pi_a^{*'}) < s(\pi_b^*, \pi_a^*) = s,$$

which contradicts market clearing. Therefore, the equilibrium is unique. ■

5.3 Solving the equilibrium

The challenging part in the equilibrium computation is solving the functional equations (21) and (22). While lemma 11 guarantees existence and uniqueness of the solution, solving these equations in practice remains non-trivial. We consider two approaches. When $\psi_1 = 1$, we transform the problem into a system of ordinary differential equations, which can be solved more efficiently. When $\psi_1 \in (0, 1)$, we use a numerical procedure that approximates the distributions. More details on the implementation are provided in appendix C.3.

Case $\psi_1 = 1$: We define the auxiliary functions

$$\beta(\delta) \equiv \frac{\gamma}{\gamma + \lambda} F(\underline{\delta}_b) + \int_{\underline{\delta}_b}^{\delta} \frac{\gamma}{\gamma + \lambda(1 - B(x))} f(x) dx \tag{26}$$

for the bid side, and

$$\alpha(\delta) \equiv \frac{\gamma}{\gamma + \lambda}(1 - F(\overline{\delta}_a)) + \int_{\delta}^{\overline{\delta}_a} \frac{\gamma}{\gamma + \lambda A(x)} f(x) dx. \tag{27}$$

for the ask side. In equilibrium, these functions satisfy

$$\beta(\delta) = \frac{\Phi_1(\delta)}{s} \quad \text{for } \delta \in \left[\underline{\delta}_b, \overline{\delta}_b\right], \qquad \alpha(\delta) = 1 - \frac{\Phi_0(\delta)}{1-s} \quad \text{for } \delta \in \left[\underline{\delta}_a, \overline{\delta}_a\right].$$

The next proposition provides a differential characterization of the bid and ask distributions in the voice trading channel.

Proposition 7. *If $\psi_1 = 1$, the unique solution B of (21) is*

$$B(\delta) = 1 - \frac{\gamma}{\lambda}\left(\frac{f(\delta)}{\beta'(\delta)} - 1\right) = 1 - \frac{\gamma}{\lambda}\left(\frac{r\pi_b^* f(\delta)}{\beta(\delta)^2 - \gamma\pi_b^* f(\delta)} - 1\right),$$

where β is the unique solution of the Riccati equation

$$\begin{cases} r\beta'(\delta) = \frac{1}{\pi_b^*}\beta(\delta)^2 - \gamma f(\delta) \\ \beta(\underline{\delta}_b) = \frac{\gamma}{\gamma+\lambda} F(\underline{\delta}_b) \end{cases}$$

for $\delta \in (\underline{\delta}_b, \overline{\delta}_b)$, where $\overline{\delta}_b = \inf\left\{x \in \left(\underline{\delta}_b, \overline{\delta}\right] : B(x) = 1\right\} \cup \left\{\overline{\delta}\right\}$.

If $\psi_1 = 1$, the unique solution A of (22) is

$$A(\delta) = -\frac{\gamma}{\lambda}\left(1 + \frac{f(\delta)}{\alpha'(\delta)}\right) = \frac{\gamma}{\lambda}\left(\frac{r\pi_a^* f(\delta)}{\alpha(\delta)^2 - \gamma\pi_a^* f(\delta)} - 1\right),$$

where α is the unique solution of the Riccati equation

$$\begin{cases} r\alpha'(\delta) = -\frac{1}{\pi_a^*}\alpha(\delta)^2 + \gamma f(\delta) \\ \alpha(\overline{\delta}_a) = \frac{\gamma}{\gamma+\lambda}(1 - F(\overline{\delta}_a)) \end{cases}$$

for $\delta \in (\underline{\delta}_a, \overline{\delta}_a)$, where $\underline{\delta}_a = \sup\left\{x \in \left[\underline{\delta}, \overline{\delta}_a\right) : A(x) = 0\right\} \cup \left\{\overline{\delta}\right\}$.

Case $\psi_1 \in (0,1)$: The proof of lemma 11 shows that solving (21) and (22) up to some $\overline{\delta}_b$ and $\underline{\delta}_a$ is essentially equivalent to solving

$$B(\delta) = \tilde{\mathbf{h}}\left\{\frac{\pi_b^*}{\nu_1}, \left(\frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\delta} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B(x))}dx\right)\right.$$
$$\left.\left(\frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \int_{\underline{\delta}_b}^{\delta} \frac{\gamma}{\gamma+\lambda\tilde{g}(1-B(x))}f(x)dx\right)\right\}$$

and

$$A(\delta) = 1 - \tilde{\mathbf{h}}\left\{\frac{\pi_a^*}{\nu_1}, \left(\frac{\pi_a^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{1-F(\overline{\delta}_a)} - \int_{\delta}^{\overline{\delta}_a} \frac{1}{r+\gamma+\lambda\tilde{g}(A(x))}dx\right)\right.$$
$$\left.\left(\frac{\gamma}{\gamma+\lambda}(1-F(\overline{\delta}_a)) + \int_{\delta}^{\overline{\delta}_a}\frac{\gamma}{\gamma+\lambda\tilde{g}(A(x))}f(x)dx\right)\right\}.$$

We provide a numerical method to approximate the unique solutions. To this end, we consider uniform partitions of $[\underline{\delta}_b, \overline{\delta}]$ and $[\underline{\delta}, \overline{\delta}_a]$ with mesh sizes $\Delta_b = \frac{\overline{\delta}-\underline{\delta}_b}{N}$ and $\Delta_a = \frac{\overline{\delta}_a - \underline{\delta}}{N}$. The partition points are $\left\{\delta_n^{(b)} = \underline{\delta}_b + n\Delta_b\right\}_{n=0}^{N}$ and $\left\{\delta_n^{(a)} = \underline{\delta} + n\Delta_a\right\}_{n=0}^{N}$. We denote $\hat{B}_n^{(N)}$ and $\hat{A}_n^{(N)}$ the approximation of $B(\delta_n^{(b)})$ and $A(\delta_n^{(a)})$. Inspired by the forward Euler method to numerically solve differential equations, these approximate values are computed recursively as follows:

$$\hat{B}_n^{(N)} \equiv \tilde{\mathbf{h}}\left\{\frac{\pi_b^*}{\nu_1}, \left(\frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} - \Delta_b\sum_{i=0}^{n-1}\frac{1}{r+\gamma+\lambda\tilde{g}(1-\hat{B}_i^{(N)})}\right)\right.$$
$$\left.\left(\frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \sum_{i=0}^{n-1}\frac{\gamma}{\gamma+\lambda\tilde{g}(1-\hat{B}_i^{(N)})}\left(F(\delta_{i+1}^{(b)}) - F(\delta_i^{(b)})\right)\right)\right\}$$

with $\hat{B}_0^{(N)} = 0$ and

$$\hat{A}_{N-n}^{(N)} \equiv 1 - \tilde{\mathbf{h}}\left\{\frac{\pi_a^*}{\nu_1}, \left(\frac{\pi_a^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{1-F(\overline{\delta}_a)} - \Delta_a\sum_{i=0}^{n-1}\frac{1}{r+\gamma+\lambda\tilde{g}(\hat{A}_{N-i}^{(N)})}\right)\right.$$
$$\left.\cdot\left(\frac{\gamma}{\gamma+\lambda}(1-F(\overline{\delta}_a)) + \sum_{i=0}^{n-1}\frac{\gamma}{\gamma+\lambda\tilde{g}(\hat{A}_{N-i}^{(N)})}\left(F(\delta_{N-i}^{(b)}) - F(\delta_{N-(i+1)}^{(b)})\right)\right)\right\}$$

with $\hat{A}_N^{(N)} = 1$. The next proposition establishes that these approximate solutions converge uniformly to the true solutions $B(\cdot|\pi_b^*)$ and $A(\cdot|\pi_a^*)$ as the partitions become finer.

Proposition 8.

$$\lim_{N\to\infty}\max_{0\le n\le N}|B(\delta_n^{(b)}|\pi_b^*) - \hat{B}_n^{(N)}| = 0$$
$$\lim_{N\to\infty}\max_{0\le n\le N}|A(\delta_n^{(a)}|\pi_a^*) - \hat{A}_n^{(N)}| = 0$$

The proof is in appendix C.3 and follows standard arguments for the convergence of the Euler method.

6 Economic implications

This section studies the economic implications of the model. We analyze how supply, dealer competition and search frictions jointly shape equilibrium allocations, and examine the resulting implications for liquidity, allocation, and welfare.

6.1 Supply and marginal type

We characterize the relationship between supply and the marginal type in equilibrium. Let $\delta^*(s)$ denote the marginal type associated with a given supply level $s \in (0, 1)$.

Lemma 13. *$\delta^*(s)$ is continuous, $\lim_{s\to 0} \delta^*(s) = \overline{\delta}$, $\lim_{s\to 1} \delta^*(s) = \underline{\delta}$, and $\delta^*(s)$ is strictly decreasing.*

The result is intuitive and in line with lemma 12. As supply increases, a larger mass of assets must be absorbed by investors, which requires lowering the marginal type $\delta^*(s)$ to include investors with lower valuations. Conversely, when supply is scarce, only high-valuation investors participate, and the marginal type converges to the upper bound of the support.

Lemma 13 allows us to express the supply as a function of the marginal type. Let $s(\delta^*)$ denote the inverse mapping, and let $B(\cdot|\delta^*)$ and $A(\cdot|\delta^*)$ be the equilibrium bid and ask distributions associated with δ^*. Using the fact that the bid and ask sides are separated at δ^*, (25) can be expressed as

$$s(\delta^*) = \frac{\chi_a(\delta^*)}{\chi_b(\delta^*) + \chi_a(\delta^*)}, \tag{28}$$

where

$$\chi_b(\delta^*) \equiv \int_{\underline{\delta}}^{\delta^*} \frac{\lambda q_b(\delta|\delta^*)}{\gamma + \lambda q_b(\delta|\delta^*)} f(\delta) dx \quad \text{and} \quad q_b(\delta|\delta^*) \equiv \tilde{g}(1 - B(\delta|\delta^*)),$$
$$\chi_a(\delta^*) \equiv \int_{\delta^*}^{\overline{\delta}} \frac{\lambda q_a(\delta|\delta^*)}{\gamma + \lambda q_a(\delta|\delta^*)} f(\delta) dx \quad \text{and} \quad q_a(\delta|\delta^*) \equiv \tilde{g}(A(\delta|\delta^*)).$$

$q_b(\delta|\delta^*)$ is the fraction of investors of type δ who accept a bid in an RFQ, and $\chi_b(\delta^*)$ captures the aggregate contribution of the bid side to supply. Similarly, $q_a(\delta|\delta^*)$ is the fraction of investors who accept an ask and $\chi_a(\delta^*)$ captures the contribution of the ask side. The representation in (28) shows that equilibrium supply is entirely pinned down by the relative contribution of the ask side to total trading activity. In equilibrium, supply reflects the balance between inflows from buyers and outflows from sellers, both of which depend on the endogenous acceptance probabilities induced by the RFQ mechanism.

The analysis of Section 5.2 implies that the bid and ask sides can be solved independently once the marginal type δ^* is fixed. In particular, $B(\cdot|\delta^*)$ depends only on the distribution F restricted to $(\underline{\delta}, \delta^*)$, while $A(\cdot|\delta^*)$ depends only on F restricted to $(\delta^*, \overline{\delta})$. As a result, the two sides are independent of each other and of the equilibrium supply s once the marginal type δ^* is fixed. Formally, for a given δ^*, the bid and ask distributions are obtained by solving (21) and (22) with the unique values π_b^* and π_a^* such that $\delta_b(\pi_b^*) = \delta^* = \delta_a(\pi_a^*)$. This separation, together with the scale invariance of the problem, motivates the introduction of the rescaled contributions

$$\tilde{\chi}_b(\delta^*) \equiv \frac{\chi_b(\delta^*)}{F(\delta^*)} \quad \text{and} \quad \tilde{\chi}_a(\delta^*) \equiv \frac{\chi_a(\delta^*)}{1 - F(\delta^*)}.$$

This normalization removes the mechanical effect of the mass of investors on each side of δ^* and isolates the relative intensity of trading induced by the curvature of F. The next proposition provides a decomposition of equilibrium supply into scale and composition effects, building on the analysis in section 2.2.

Proposition 9. *The equilibrium supply admits the decomposition*

$$\frac{1 - s(\delta^*)}{s(\delta^*)} = \frac{F(\delta^*)}{(1 - F(\delta^*))} \frac{\tilde{\chi}_b(\delta^*)}{\tilde{\chi}_a(\delta^*)}.$$

Proposition 9 provides a tractable decomposition of equilibrium supply into scale and composition effects. The scale effect is captured by $\frac{F(\delta^*)}{1-F(\delta^*)}$, while the composition effect is captured by the ratio $\frac{\tilde{\chi}_b(\delta^*)}{\tilde{\chi}_a(\delta^*)}$. The scale effect reflects the relative mass of investors on each side of the marginal type. When δ^* is high, the mass of potential buyers with $\delta > \delta^*$ is small, which limits the ability of the market to absorb supply. Conversely, when δ^* is low, a large mass of investors is available to purchase the asset, allowing for higher supply.

The composition effect captures how the composition of trading opportunities differs across the two sides of the market. In an equivalent model without private information, all investors with $\delta < \delta^*$ sell and all investors with $\delta > \delta^*$ buy, upon meeting dealers. In this case, the fraction of investors who trade on each side is degenerate, with $q_b(\delta|\delta^*) = \mathbf{1}_{\delta<\delta^*}$ and $q_a(\delta|\delta^*) = \mathbf{1}_{\delta>\delta^*}$, so that $\tilde{\chi}_b(\delta^*) = \tilde{\chi}_a(\delta^*) = \frac{\lambda}{\gamma+\lambda}$. The composition effect disappears because the contributions of the bid and ask sides are identical and the supply is entirely determined by the scale effect, which yields the standard relation $1 - s = F(\delta^*)$.

With private information, the fractions of investors who trade on the bid and ask sides are no longer degenerate. Instead, they depend on the price distributions B and A, which themselves depend on the distribution of types F on each side of δ^*. As a result, aggregate supply depends not only on the relative mass of investors on each side of δ^*, but also on the relative quality of trading opportunities generated by the curvature of F on the left and right of the cutoff. When the bid side is relatively more favorable to investors than the ask side, a larger relative fraction of low-valuation investors sell than high-valuation investors buy. This raises the relative contribution of the bid side and therefore increases $\frac{1-s(\delta^*)}{s(\delta^*)}$, so that supply is lower than in the benchmark. Conversely, when the ask side is relatively more favorable, a larger fraction of high-valuation investors buy than low-valuation investors sell, and supply is higher than in the benchmark.

To further characterize the composition effect, we introduce normalized distributions on each side of the cutoff, following the approach of Section 2.2. We define

$$\tilde{F}_b(z|\delta^*) \equiv \frac{F(\underline{\delta} + z(\delta^* - \underline{\delta}))}{F(\delta^*)} \quad \text{and} \quad \tilde{F}_a(z|\delta^*) \equiv \frac{F(\delta^* + z(\overline{\delta} - \delta^*)) - F(\delta^*)}{1 - F(\delta^*)}$$

for $z \in [0,1]$. The distributions $\tilde{F}_b(\cdot|\delta^*)$ and $\tilde{F}_a(\cdot|\delta^*)$ correspond to F restricted to each side of δ^* and rescaled to $[0,1]$. The next proposition shows how their variation with δ^* governs the composition effect.

Proposition 10. *If $\tilde{F}_b(\cdot \mid \delta^*)$ is decreasing (increasing) in δ^* with respect to FOSD, then $\tilde{\chi}_b(\delta^*)$ is decreasing (increasing). The monotonicity is strict if the FOSD shift in $\tilde{F}_b$ is strict.*

If $\tilde{F}_a(\cdot \mid \delta^*)$ is decreasing (increasing) in δ^* with respect to FOSD, then $\tilde{\chi}_a(\delta^*)$ is increasing (decreasing). The monotonicity is strict if the FOSD shift in $\tilde{F}_a$ is strict.

The normalized distributions $\tilde{F}_b(\cdot|\delta^*)$ and $\tilde{F}_a(\cdot|\delta^*)$ describe the composition of investors on each side of the cutoff, abstracting from scale. Changes in δ^* shift these distributions and therefore alter the composition of trading opportunities faced by dealers. On the bid side, if $\tilde{F}_b(\cdot|\delta^*)$ shifts downward in the FOSD sense as δ^* increases, the pool of potential sellers becomes less favorable, which reduces the intensity of trading and lowers $\tilde{\chi}_b(\delta^*)$. Conversely, if the distribution shifts upward, the composition improves and trading becomes more intense. On the ask side, the effect is reversed. A downward shift in $\tilde{F}_a(\cdot|\delta^*)$ improves the composition of potential buyers, increasing trading intensity and raising $\tilde{\chi}_a(\delta^*)$, while an upward shift worsens it.

This result connects naturally with lemma 3. If F follows a Beta distribution with parameters $a, b \geq 1$, both $\tilde{F}_b(\cdot \mid \delta^*)$ and $\tilde{F}_a(\cdot \mid \delta^*)$ are decreasing in δ^* with respect to FOSD. As a result, $\tilde{\chi}_b(\delta^*)$ decreases while $\tilde{\chi}_a(\delta^*)$ increases. Therefore, the composition effect reinforces the scale effect: as δ^* increases, the relative contribution of the ask side rises compared to the bid side, leading to higher equilibrium supply. A particularly transparent case is the Beta$(a, 1)$ distribution. In this case, $\tilde{F}_b(\cdot \mid \delta^*)$ does not vary with δ^*, so the composition effect is absent on the bid side and $\tilde{\chi}_b(\delta^*)$ remains constant. By contrast, $\tilde{F}_a(\cdot \mid \delta^*)$ is decreasing in δ^* with respect to FOSD, which implies that $\tilde{\chi}_a(\delta^*)$ is increasing. As a result, changes in supply relative to the benchmark are entirely driven by the ask side: as δ^* increases, the composition of buyers improves, raising the relative contribution of the ask side and increasing equilibrium supply.

Another interesting benchmark is the uniform distribution. In this case, the normalized distributions $\tilde{F}_b(\cdot \mid \delta^*)$ and $\tilde{F}_a(\cdot \mid \delta^*)$ do not depend on δ^*, so there is no composition effect. As a result, $\tilde{\chi}_b(\delta^*) = \tilde{\chi}_a(\delta^*) = \chi$, and the supply is entirely determined by the scale effect. Consequently, the equilibrium condition reduces to the benchmark relation $1 - s(\delta^*) = F(\delta^*)$. The intuition is that the uniform distribution has a flat density, so the composition of investors on each side of the cutoff is invariant to changes in δ^*. Therefore, private information does not distort the relative quality of trading opportunities across the two sides of the market, and the equilibrium allocation coincides with the frictionless benchmark.

Figure 6 illustrates how the composition effect translates into changes in marginal type relative to the benchmark without private information and confirms the economic intuition. The figure also considers the case $F = \text{Beta}(2, 2)$, which is symmetric. In this case, the rescaled distributions on the bid and ask sides coincide at $\delta^* = 0.5$, so that the composition effect vanishes when computing the corresponding supply. As a result, $s(\delta^* = 0.5) = 0.5$ coincides with the benchmark value. The boundary conditions $\lim_{s \to 0} \delta^*(s) = \underline{\delta}$ and $\lim_{s \to 1} \delta^*(s) = \overline{\delta}$ imply that the scale effect necessarily dominates in these extreme regimes.

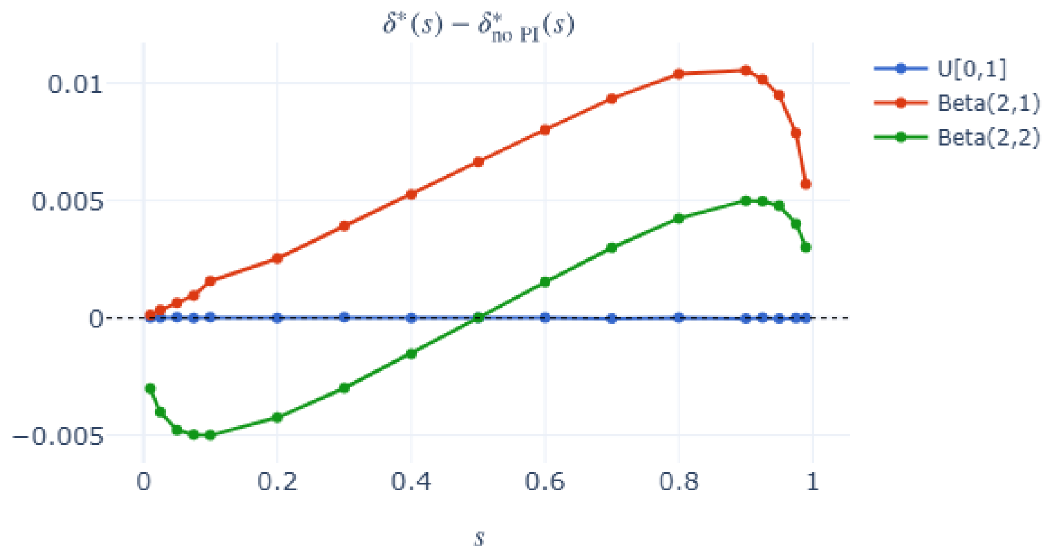

Figure 6: Difference between the marginal type $\delta^*(s)$ and the benchmark without private information $\delta^*_{\text{no PI}}(s) = F^{-1}(1-s)$ as a function of s, for $(r, \gamma, \lambda) = (0.05, 1, 1)$ and $\Psi \sim \text{NB}(3.75, 0.9)|_{>0}$. When $F = U[0,1]$, composition effects cancel and δ^* coincides with its benchmark. When $F = \text{Beta}(2,1)$, composition effects arise only on the ask side, raising δ^* relative to the benchmark. When $F = \text{Beta}(2,2)$, composition effects operate on both sides of the market but are symmetric at $\delta^* = s = 0.5$.

6.2 Dealer competition

We now study the effects of dealer competition on equilibrium outcomes. Throughout this section, we compare two participation distributions Ψ and Ψ', and assume that $\Psi' \succ \Psi$, that is, Ψ' corresponds to a more competitive environment for dealers. Accordingly, we index equilibrium objects by Ψ or Ψ' to indicate the environment under consideration.

We first contrast the findings of proposition 4 from section 2.2 of the static RFQ model, where increasing competition leads to a strict improvement in quotes in the sense of FOSD. In this dynamic setting, this monotonicity no longer holds when quotes are expressed in type space. Figure 7 illustrates the bid and ask distributions for different levels of competition. Focusing on the bid side, we observe that higher competition can generate regions in which prices are less favorable for investors in type space, that is, where $B(\delta|\Psi') > B(\delta|\Psi)$. A symmetric observation holds on the ask side where $A(\delta|\Psi') < A(\delta|\Psi)$ on some regions. In fact, as figure 7 suggests, the next proposition shows that the supports of B and A expand on both ends as competition increases.

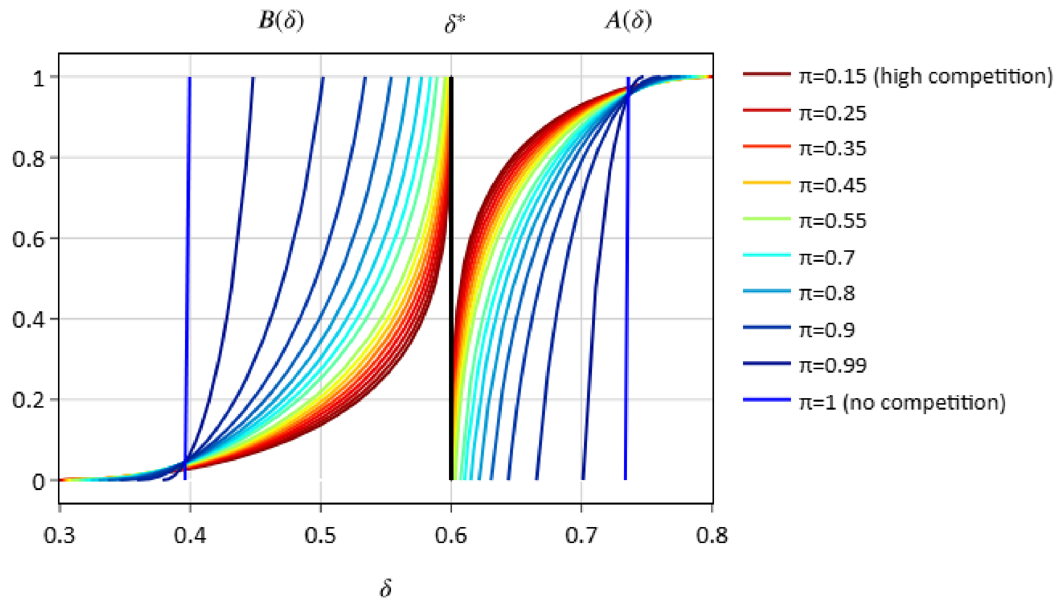

Figure 7: Price distributions for different levels of competition, for $(r, \gamma, \lambda, s) = (0.05, 1, 1, 0.4)$, $\Psi \sim \text{NB}(3.75, \pi)|_{>0}$, and $F = U[0, 1]$. Increasing competition does not uniformly improve prices in type space for investors due to general equilibrium feedback effects. In particular, greater competition expands the support on both sides of the distributions. As a result, some investors who would always accept quotes under low competition may optimally refuse them when competition increases.

Proposition 11. *Increasing competition, $\Psi' \succ \Psi$, strictly expands the supports of the bid and ask distributions on both ends,*

$$\underline{\delta}_b(\Psi') < \underline{\delta}_b(\Psi) \quad \textit{and} \quad \overline{\delta}_b(\Psi') > \overline{\delta}_b(\Psi),$$
$$\underline{\delta}_a(\Psi') < \underline{\delta}_a(\Psi) \quad \textit{and} \quad \overline{\delta}_a(\Psi') > \overline{\delta}_a(\Psi).$$

In particular, defining bid and ask price dispersion as $\overline{\delta}_b - \underline{\delta}_b$ and $\overline{\delta}_a - \underline{\delta}_a$, the proposition implies that both are increasing in competition. One side of the result is intuitive: more competition induces dealers to offer better prices, which explains the increase in $\overline{\delta}_b$ and the decrease in $\overline{\delta}_a$. The expansion of the support in the opposite direction is less immediate and reflects general equilibrium feedback effects. First, better trading conditions compress differences in surplus $|P - R(\delta)|$ across types. This reduces realized profits of dealers and induces them to target lower types on the bid side or to target higher types on the ask side in order to recover rents.

Second, improved trading conditions affect the endogenous distributions of asset allocation Φ_1 and Φ_0. In particular, improved trading conditions increase allocation turnover on the ask side, which raises the mass of investors holding the asset. Since these investors are subject to type resets while holding the asset, the composition of sellers becomes more sensitive to the reset process and less tied to their initial trading conditions. This increases the mass of low valuation asset holders and reshapes the distribution Φ_1. As a result, dealers have stronger incentives to target low valuation investors. A symmetric mechanism operates on the ask side. These two effects reinforce each other and lead dealers to compete for a wider range of types, which expands the supports of B and A in both directions. Competition reshapes not only prices, but the endogenous adverse selection environment faced by dealers

6.3 Search frictions and type dynamics

We now study the role of search frictions and type dynamics in shaping equilibrium outcomes. In particular, we focus on the limiting regimes in which $\lambda \to \infty$ and $\gamma \to \infty$. These limits isolate the roles of frictions and heterogeneity, and connect the dynamic equilibrium to tractable benchmark environments. We index equilibrium objects by λ or γ whenever we want to emphasize their dependence on the meeting rate or the type reset rate.

Proposition 12. *As the meeting rate λ tends to infinity, both the bid and ask distributions collapse to a degenerate distribution at the marginal type $\delta^* = F^{-1}(1-s)$,*

$$\lim_{\lambda\to\infty} B(\delta|\lambda) = \mathbf{1}_{\delta\geq\delta^*} \quad \textit{and} \quad \lim_{\lambda\to\infty} A(\delta|\lambda) = \mathbf{1}_{\delta\geq\delta^*},$$

for all $\delta \in \mathcal{D}\backslash\{\delta^*\}$.

As λ becomes large, investors can meet dealers almost instantaneously and therefore have immediate access to their outside option. This eliminates the intermediation friction that gives dealers market power. In this limit, dealers effectively compete à la Bertrand: any attempt to extract rents is undercut by competing dealers, driving profits to zero. As a result, quotes must coincide with the interdealer price, which pins down a unique trading threshold. This eliminates price dispersion and leads both the bid and ask distributions to collapse to a degenerate distribution at the marginal type

δ^*. In equilibrium, this yields the standard threshold allocation $F(\delta^*) = 1 - s$. Figure 8 shows how the bid and ask distributions concentrate around δ^* as λ increases.

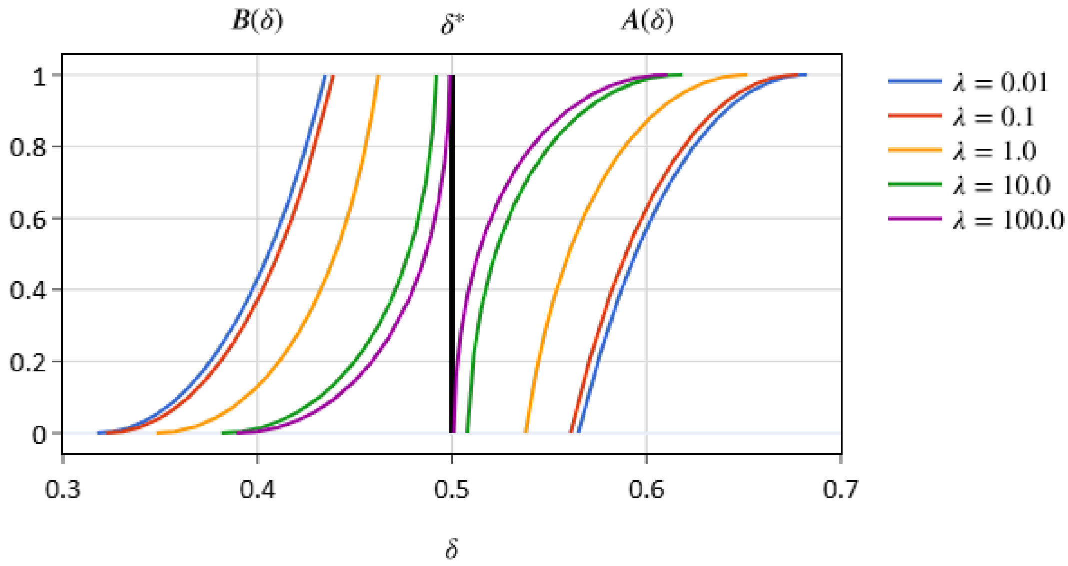

Figure 8: Price distributions for different levels of search intensity λ, when $(r, \gamma, s) = (0.05, 1, 0.5)$, $\Psi \sim \text{NB}(3.75, 0.9)|_{>0}$, and $F = \text{Beta}(2, 2)$. As the trading intensity increases, both the bid and ask price distributions converge to degenerate distributions at δ^*. The limit $\lambda \to \infty$ corresponds to Bertrand competition, as investors can immediately obtain alternative quotes from other dealers.

Proposition 13. *As the reset rate γ tends to infinity, both the bid and ask distributions converge to the static solution of the RFQ auction with an interdealer price $\delta^* \in (\underline{\delta}, \overline{\delta})$.*

$$\lim_{\gamma \to \infty} B(\delta|\gamma) = B_{static}(\delta) \quad \textit{and} \quad \lim_{\gamma \to \infty} A(\delta|\gamma) = A_{static}(\delta),$$

for all $\delta \in \mathcal{D}$. $B_{static}(\delta)$ and $A_{static}(\delta)$ denote the solution of the RFQ auction of section 2 when $P = \delta^*$ and $\Phi_1 = \Phi_0 = F$. In particular, when $\psi_1 = 1$, both $B_{static}(\delta)$ and $A_{static}(\delta)$ are degenerate distributions, concentrated at a point in $(\underline{\delta}, \delta^*)$ and $(\delta^*, \overline{\delta})$ respectively.

To understand this result, lemmas 4 and 5 show that as $\gamma \to \infty$, $R(\delta) = R(\delta^*)$, $\Phi_1(\delta) = sF(\delta)$, and $\Phi_0(\delta) = (1 - s)F(\delta)$ for all $\delta \in \mathcal{D}$. The reservation value is independent of δ, which reflects the fact that the types of investors are reshuffled so frequently that future payoffs no longer depend on their current valuation. $\Phi_1(\delta) = sF(\delta)$ reflects the fact that asset holders are continuously reshuffled, so that the composition of holders mirrors the overall population distribution F, and similarly for non-holders. The dynamic feedback between trading and type composition disappears.

A first-order expansion of the reservation value around δ^* yields

$$R(\delta) \approx R(\delta^*) + R'(\delta^*)(\delta - \delta^*),$$

so that monopolistic profits take the form

$$\hat{\Pi}_b(\delta) \approx R'(\delta^*)(\delta^* - \delta)F(\delta) \quad \text{and} \quad \hat{\Pi}_a(\delta) \approx R'(\delta^*)(\delta - \delta^*)(1 - F(\delta)).$$

These expressions coincide with the static RFQ problem times a constant, which explains why the bid and ask distributions converge to their static counterparts. In economic terms, when types are reshuffled extremely frequently, the endogenous feedback between trading, reservation values and the distribution of types disappears. This result is illustrated in figure 9.

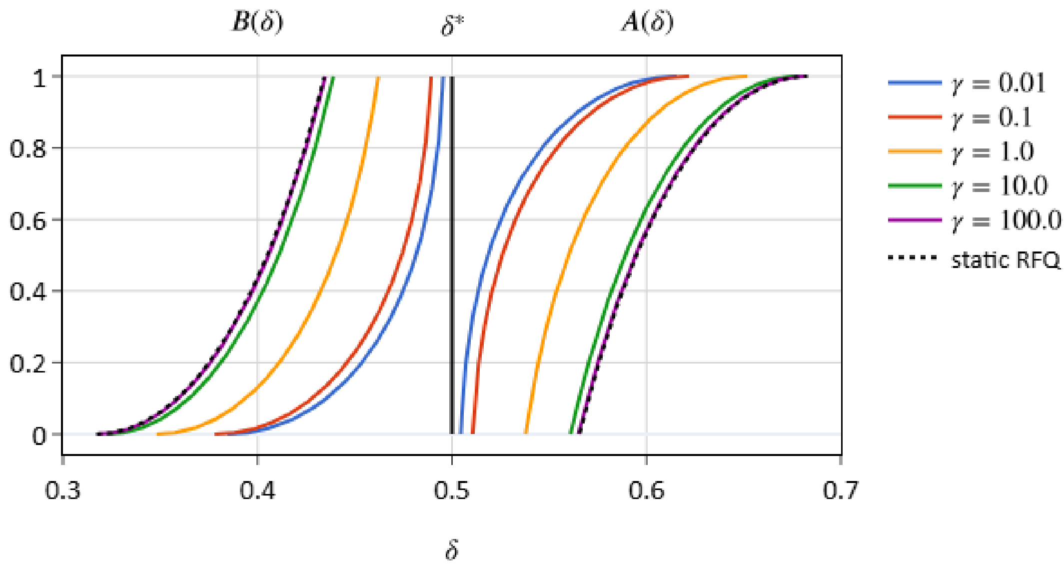

Figure 9: Price distributions for different levels of reset intensity γ, when $(r, \lambda, s) = (0.05, 1, 0.5)$, $\Psi \sim \text{NB}(3.75, 0.9)|_{>0}$, and $F = \text{Beta}(2, 2)$. The static solution of the auction is also displayed. As γ increases, the endogenous general equilibrium feedback vanishes, and the RFQ auction converges to its static counterpart.

6.4 Liquidity, allocation and welfare

7 Quantitative analysis

This section provides a quantitative illustration of the model. We first derive a closed-form solution in a tractable specification corresponding to the voice trading channel with a uniform distribution of types, which makes some of the main mechanisms transparent. We then present a simple calibration of the RFQ model with dealer competition and a Beta distribution of types to assess the quantitative implications.

7.1 A closed-form solution

We consider the voice trading channel with $\psi_1 = 1$ and assume a uniform distribution of types $F \sim U[0,1]$. In this configuration, the Riccati system in proposition 7 admits a closed-form solution, which yields a fully explicit characterization of the equilibrium. To express this characterization, we define the constants

$$\xi_0 \equiv \frac{2}{r}\sqrt{\gamma(r+\gamma)} \qquad \xi_\lambda \equiv \frac{2}{r}\sqrt{(\gamma+\lambda)(r+\gamma+\lambda)},$$

and

$$\eta_0 \equiv \log\left(1+\frac{2}{r}\gamma-\xi_0\right), \qquad \eta_\lambda \equiv \log\left(1+\frac{2}{r}(\gamma+\lambda)+\xi_\lambda\right).$$

Proposition 14. *If $\psi_1 = 1$ and $F \sim U[0,1]$, the equilibrium price distributions are given by*

$$B(\delta) = \frac{r}{2\lambda}(e^{\eta_\lambda}-\xi_\lambda) - \frac{r}{2\lambda}\cosh\left(\eta_\lambda - \xi_\lambda\frac{\delta-\underline{\delta}_b}{\underline{\delta}_b}\right) \qquad \text{for all } \delta \in \left[\underline{\delta}_b, \overline{\delta}_b\right]$$

and

$$A(\delta) = \frac{r}{2\lambda}\cosh\left(\eta_\lambda + \xi_\lambda\frac{\overline{\delta}_a-\delta}{1-\overline{\delta}_a}\right) - \frac{r}{2\lambda}(e^{\eta_0}+\xi_0) \qquad \text{for all } \delta \in \left[\underline{\delta}_a, \overline{\delta}_a\right],$$

where the thresholds satisfy

$$\delta^* = 1 - s,$$

$$\underline{\delta}_b = \frac{(1-s)\xi_\lambda}{\eta_0 + \eta_\lambda + \xi_0 + \xi_\lambda}, \qquad \overline{\delta}_b = \frac{(1-s)(\eta_0 + \eta_\lambda + \xi_\lambda)}{\eta_0 + \eta_\lambda + \xi_0 + \xi_\lambda},$$

$$\underline{\delta}_a = 1 - \frac{s(\eta_0 + \eta_\lambda + \xi_\lambda)}{\eta_0 + \eta_\lambda + \xi_0 + \xi_\lambda}, \qquad \overline{\delta}_a = 1 - \frac{s\xi_\lambda}{\eta_0 + \eta_\lambda + \xi_0 + \xi_\lambda}.$$

The proof is provided in Hugonnier et al. (2025). These expressions yield explicit formulas for the equilibrium allocations Φ_1 and Φ_0, the interdealer price $P = R(\delta^*)$, and the reservation value R, which are reported in Appendix E. Taken together, they provide a complete closed-form characterization of equilibrium allocations and prices in this benchmark environment.

A first observation is that the marginal type satisfies $\delta^* = 1 - s$, which confirms the result obtained in section 6.1. In particular, the bid and ask sides are symmetric after rescaling, and their relative contributions to aggregate supply are symmetric. We also confirm the results of section 6.3, since a direct calculation shows that $\lim_{\lambda \to 0}(\overline{\delta}_b - \underline{\delta}_b) = \lim_{\gamma \to 0}(\overline{\delta}_b - \underline{\delta}_b) = 0$. When trading opportunities become frequent, dealer competition intensifies and drives quotes toward the interdealer price. When type resets become frequent, the composition of types becomes independent of trading conditions, eliminating the general equilibrium feedback loop. In both limit cases, price dispersion collapses.

The closed-form expressions for B and A in Proposition 14 show that the curvature of the price distributions is governed by the parameters ξ_λ and η_λ through their appearance in the hyperbolic cosine terms. In particular, the slope can become steep very quickly. This is the case for empirically plausible parameter values such as $r = 0.05$, $\gamma \in [0.1, 10]$, and $\lambda \in [0.1, 10]$. For such parameters, the maximal relative support size satisfies

$$\max \frac{\overline{\delta}_b - \underline{\delta}_b}{\delta^*} = \max \frac{\overline{\delta}_a - \underline{\delta}_a}{1 - \delta^*} \approx 5.3\%.$$

Thus, while the closed-form solution is useful for illustrating some of the mechanisms of the model, it generates quantitatively limited price dispersion. To obtain richer dispersion patterns, it is necessary to move beyond the uniform benchmark and allow for more general type distributions and dealer competition at the pricing stage.

7.2 A simple calibration

8 Conclusion

In this paper, we develop a theory of dealer intermediation in OTC markets with search frictions and investor private valuations. We embed an RFQ trading protocol into a general equilibrium framework and we show that prices, allocations, and trading outcomes are jointly determined by supply and demand. This interaction generates novel asymmetries between the bid and ask sides of the market and leads to endogenous price elasticities.

Our analysis highlights two key mechanisms. First, adverse selection operates through both scale and composition effects, which shape how prices and allocations respond to changes in market conditions. Second, dealer competition has non-trivial equilibrium effects: it worsens prices in type space for some investors and expands price dispersion at both ends of the distributions.

On the methodological side, we establish existence and uniqueness of the equilibrium and provide a tractable characterization that can be solved in closed form in a benchmark case or numerically. Overall, the framework offers a flexible and tractable foundation for studying OTC markets with private valuation, and provides a basis for quantitative and empirical investigations of price formation, competition, and liquidity provision in decentralized markets.

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A RFQ auction proofs

A.1 Auction solution proofs

Lemma 1.

Proof. Assume towards a contradiction that $B(b) - B(b^-) > 0$ for some $b \in [\underline{r}, P)$. In particular, for all $k > 1$,

$$\begin{aligned}
&\sum_{m=0}^{k-1} \frac{1}{1+m}\binom{k-1}{m}(B(b) - B(b^-))^m B(b^-)^{k-1-m} \\
&= \sum_{m=0}^{k-1}\left(1 - (1 - \frac{1}{1+m})\right)\binom{k-1}{m}(B(b) - B(b^-))^m B(b^-)^{k-1-m} \\
&= \sum_{m=0}^{k-1}\binom{k-1}{m}(B(b) - B(b^-))^m B(b^-)^{k-1-m} \\
&\underbrace{- \sum_{m=1}^{k-1}\left(1 - \frac{1}{1+m}\right)\binom{k-1}{m}(B(b) - B(b^-))^m B(b^-)^{k-1-m}}_{L_k} \\
&= B(b)^{k-1} - L_k
\end{aligned}$$

where $L_k > 0$. Let $\epsilon_n > 0$ be a sequence such that $\lim_{n\to\infty} \epsilon_n = 0$ and such that $B(b + \epsilon_n) = B((b + \epsilon_n)^-)$. We have

$$\begin{aligned}
&\lim_{n\to\infty} B(b + \epsilon_n)^{k-1} = B(b)^{k-1} \\
&> B(b)^{k-1} - L_k = \sum_{m=0}^{k-1} \frac{1}{1+m}\binom{k-1}{m}(B(b) - B(b^-))^m B(b^-)^{k-1-m}.
\end{aligned}$$

Moreover, if $\psi_1 < 1 \Leftrightarrow \nu_1 < 1$, there exists $k > 1$ such that $\nu_k > 0$ and

$$\lim_{n\to\infty} \Pi_b(b+\epsilon_n|B) = (P-b)\Phi_1(b)h(B(b)) > \Pi_b(b),$$

which contradicts b maximizing $\Pi_b(\cdot|B)$. For the remainder of the proof, we can thus consider (4) as the expression of $\Pi_b(b|B)$.

If $\psi_1 \in (0,1) \Leftrightarrow \nu_1 \in (0,1)$ and $P > \underline{r}$, $\Pi_b(b|B) > 0$ in a right neighbourhood of $\underline{r}$ since $h(0) = \nu_1 > 0$. This implies that $P \notin \text{supp}(B)$ and proves the result. If $\psi_1 = 0 \Leftrightarrow \nu_1 = 0$, we define $\underline{r}_b \equiv \inf \text{supp}(B)$. Assume towards a contradiction that $\underline{r}_b < P$. The continuity of B implies that

$$\lim_{b\downarrow \underline{r}_b} \Pi_b(b|B) = (P-\underline{r}_b)\Phi_1(\underline{r}_b^-)h(0) = 0,$$

and thus $\sup_{b\in[\underline{r},P]} \Pi_b(b|B) = 0$. However, $\Pi_b(b|B) > 0$ for all $b \in (\underline{r}_b, P)$. The proof for the ask side is similar. ■

Lemma 14. *$\hat{\Pi}_b^*(b)$ is continuous, decreasing and the supremum is attained for all $b \in [\underline{r}, P]$. $\hat{\Pi}_a^*(a)$ is continuous, increasing and the supremum is attained for all $a \in [P, \bar{r}]$.*

Proof. If $b \in [\underline{r}, P]$, then $\hat{\Pi}_b^*(b) \equiv \sup_{x\in[b,P]} \hat{\Pi}_b(x)$. The decrease of $\hat{\Pi}_b^*$ follows directly from the definition. To show that the supremum is attained, we observe that $\hat{\Pi}_b$ is bounded and right-continuous. In particular, $\hat{\Pi}_b(b) \geq \hat{\Pi}_b(b^-)$ since

$$\hat{\Pi}_b(b) - \hat{\Pi}_b(b^-) = (P-b)(\Phi_1(b) - \Phi_1(b^-)) \geq 0. \tag{29}$$

Let $\epsilon_n > 0 = \lim_{n\to\infty} \epsilon_n$ and let $b_n \in [b, P]$ such that $\hat{\Pi}_b^*(b) \geq \hat{\Pi}_b(b_n) \geq \hat{\Pi}_b^*(b) - \epsilon_n$. In particular, $\lim_{n\to\infty} \hat{\Pi}_b(b_n) = \hat{\Pi}_b^*(b)$. There exists a converging subsequence n_l such that $\lim_{l\to\infty} b_{n_l} = \hat{b} \in [b, P]$. If there exists a converging subsequence l_j such that $b_{n_{l_j}} \geq \hat{b}$, then $\hat{\Pi}_b(\hat{b}) = \hat{\Pi}_b^*(b)$ by right-continuity. Otherwise, there exists $L \in \mathbf{N}$ such that $b_{n_l} < \hat{b}$ for all $l > L$, in which case:

$$\lim_{l\to\infty} \hat{\Pi}_b(b_{n_l}) = \hat{\Pi}_b^*(b) = \hat{\Pi}_b(\hat{b}^-) \leq \hat{\Pi}_b(\hat{b}) \leq \hat{\Pi}_b^*(b).$$

Both cases imply that $\hat{\Pi}_b(\hat{b}) = \hat{\Pi}_b^*(b)$. To show the continuity of $\hat{\Pi}_b^*$, let $b_n \in [\underline{b}, b)$ be an increasing sequence such that $\lim_{n\to\infty} b_n = b$. Denote $\hat{b}_n \in \text{argmax}_{x\in[b_n,P]} \hat{\Pi}_b(x)$. $\hat{\Pi}_b^*(b_n)$ is decreasing and $\hat{\Pi}_b^*(b_n) \geq \hat{\Pi}_b^*(b)$, therefore $\lim_{n\to\infty} \hat{\Pi}_b^*(b_n) = L \geq \hat{\Pi}_b^*(b)$. We assume towards a contradiction that $L > \hat{\Pi}_b^*(b)$, which implies that

$$\hat{\Pi}_b^*(b_n) = \hat{\Pi}_b(\hat{b}_n) \geq L > \hat{\Pi}_b^*(b)$$

and $\hat{b}_n \in [b_n, b)$. Therefore, $\lim_{n\to\infty} \hat{b}_n = b$ and $\hat{\Pi}_b(b^-) \geq L > \hat{\Pi}_b^*(b) \geq \hat{\Pi}_b(b)$, which contradicts (29), from which we conclude that $\hat{\Pi}_b^*(b^-) = \hat{\Pi}_b^*(b)$. The argument for $\hat{\Pi}_b^*(b^+) = \hat{\Pi}_b^*(b)$ is similar. The proof for $\hat{\Pi}_a^*$ is similar. ■

Proposition 1.

Proof. We first show that B is a continuous distribution with values $[\underline{r}, P)$. To do that, we simply show that B is continuous, increasing with values in $[0, 1]$. Indeed $B(\underline{r}) = h^{-1}(\nu_1) = 0$ and $B(b) = 1$ for all $b \geq P$ since $\hat{\Pi}_b^*(b) \leq 0 < \nu_1 \hat{\Pi}_b^*(\underline{r})$. Moreover, B is continuous and increasing, since lemma 14 implies that $\hat{\Pi}_b^*(\underline{r})/\hat{\Pi}_b^*(b)$ is continuous and increasing in $[\underline{r}, P]$. $\text{supp}(B) \subset [\underline{r}, P)$, since by continuity there exists $\epsilon > 0$ such that $\hat{\Pi}_b^*(b) < \nu_1 \hat{\Pi}_b^*(\underline{r})$ for all $b \in [P - \epsilon, P]$ (as $\hat{\Pi}_b^*(P) = 0$ and $\nu_1 \hat{\Pi}_b^*(\underline{r}) > 0$).

We now verify that B is an equilibrium. $\Pi_b(b|B) = \hat{\Pi}_b(b)h(B(b)) = \nu_1 \hat{\Pi}_b^*(\underline{r})$ for all $b \in \text{supp}(B)$ by construction. It only remains to show that $\Pi_b(b|B) \leq \nu_1 \hat{\Pi}_b^*(\underline{r})$ for all $b \in \mathcal{R}$. Indeed, if $\hat{\Pi}_b^*(b) < \nu_1 \hat{\Pi}_b^*(\underline{r})$ then

$$\Pi_b(b|B) = \hat{\Pi}_b(b)h(B(b)) = \hat{\Pi}_b(b)h(1) = \hat{\Pi}_b(b) \leq \hat{\Pi}_b^*(b) < \nu_1 \hat{\Pi}_b^*(\underline{r})$$

by assumption. If $\hat{\Pi}_b^*(b) \geq \nu_1 \hat{\Pi}_b^*(\underline{r})$ then

$$\Pi_b(b|B) = \hat{\Pi}_b(b)\nu_1 \hat{\Pi}_b^*(\underline{r})/\hat{\Pi}_b^*(b) \leq \hat{\Pi}_b(b)\nu_1 \hat{\Pi}_b^*(\underline{r})/\hat{\Pi}_b(b) = \nu_1 \hat{\Pi}_b^*(\underline{r}).$$

Finally we show that B is the unique equilibrium. Let B be a solution of (3). $\Pi_b(b|B) = \hat{\Pi}_b(b)h(B(b))$ implies that $B(b) = h^{-1}(\Pi_b(b|B)/\hat{\Pi}_b(b))$ for all $b \in (\underline{r}, P)$. This in turn implies that for all $b \in \text{supp}(B)$

$$B(b) = h^{-1}(\pi_b^*/\hat{\Pi}_b(b)), \tag{30}$$

where $\pi_b^* = \max_{x\in[\underline{r},P]} \Pi_b(x|B)$. Moreover, if $b \in \text{supp}(B)$ and $b' > b$, the monotonicity of h and B, and the condition $b \in \text{argmax}_{x\in[\underline{r},P]} \Pi_b(x|B)$ imply that $\hat{\Pi}_b(b) \geq \hat{\Pi}_b(b')$. If in addition, $b' \in \text{supp}(B)$, then $B(b') > B(b)$, $\Pi_b(b|B) = \Pi_b(b'|B)$ and the strict monotonicity of h imply that $\hat{\Pi}_b(b) > \hat{\Pi}_b(b')$. Combining this with (30), lemma 1 and lemma 14 shows that

$$B(b) = h^{-1}(\pi_b^* / \hat{\Pi}_b^*(b))$$

for all b such that $\pi_b^*/\hat{\Pi}_b^*(b) \in [\nu_1, 1]$. In this configuration, $\inf \operatorname{supp}(B) = \underline{r}_b$ verifies $\pi_b^* = \nu_1 \hat{\Pi}_b^*(\underline{r}_b)$. We assume towards a contradiction that $\hat{\Pi}_b^*(\underline{r}) > \hat{\Pi}_b^*(\underline{r}_b)$. Lemma 14 implies that there exists $x \in (\underline{r}, \underline{r}_b)$ such that $\hat{\Pi}_b(x) > \hat{\Pi}_b^*(\underline{r}_b)$. In particular $\Pi_b(\underline{r}_b|B) = \nu_1 \hat{\Pi}_b(\underline{r}_b) < \Pi_b(x|B) = \nu_1 \hat{\Pi}_b(x)$, which contradicts $\underline{r}_b$ maximizing $\Pi_b(\cdot|B)$. Therefore $\pi_b^* = \nu_1 \hat{\Pi}_b^*(\underline{r})$. The proof is similar for A.

■

A.2 Auction equilibrium structure proofs

Lemma 15. *If F admits a log-concave density f, then:*

- *f is absolutely continuous on $(\underline{\delta}, \overline{\delta})$,*
- *f is bounded,*
- *for all $p \in [0,1]$, $\log|F(\delta) - p|$ is concave, and $\frac{f(\delta)}{F(\delta)-p}$ is decreasing on $\delta \in (F^{-1}(p), \overline{\delta})$ and on $\delta \in (\underline{\delta}, F^{-1}(p))$.*

Proof. $f(\delta) = \exp(\log(f(\delta))$. Since $\log(f)$ is concave on $(\underline{\delta}, \overline{\delta})$, it is absolutely continuous on $(\underline{\delta}, \overline{\delta})$, and so is f.

To show the boundedness of f, we show that $\log(f)$ is bounded above. Assume towards a contradiction that this is not the case. Then there exists a sequence $\delta_k \in (\underline{\delta}, \overline{\delta})$ such that $\lim_{k\to\infty} \delta_k = \delta_0$ and $\lim_{k\to\infty} \log(f(\delta_k)) = \infty$. It is clear by continuity that $\delta_0 = \underline{\delta}$ or $\delta_0 = \overline{\delta}$. If $\delta_0 = \underline{\delta}$, for any $\hat{\delta} > \tilde{\delta} \in (\underline{\delta}, \overline{\delta})$ and k sufficiently high

$$\begin{aligned}
&\infty > \log(f(\tilde{\delta})) = \log f\left(\frac{\tilde{\delta} - \delta_k}{\hat{\delta} - \delta_k}\hat{\delta} + \frac{\hat{\delta} - \tilde{\delta}}{\hat{\delta} - \delta_k}\delta_k\right) \\
&\geq \frac{\tilde{\delta} - \delta_k}{\hat{\delta} - \delta_k}\log(f(\hat{\delta})) + \frac{\hat{\delta} - \tilde{\delta}}{\hat{\delta} - \delta_k}\log(f(\delta_k)) \\
&\geq -|\log(f(\hat{\delta}))| + \frac{\hat{\delta} - \tilde{\delta}}{\hat{\delta} - \underline{\delta}}\log(f(\delta_k)) \to_{k\to\infty} \infty,
\end{aligned}$$

which is a contradiction. A similar contradiction is obtained if $\delta_0 = \overline{\delta}$.

For the last assertion, we observe that for all $\delta \in (F^{-1}(p), \overline{\delta})$

$$\frac{d}{d\delta}\log(F(\delta)-p) = \frac{f(\delta)}{F(\delta)-p} = \frac{f(\delta)}{F(\delta)}\left(1+\frac{p}{F(\delta)-p}\right),$$

which is decreasing since $\frac{f(\delta)}{F(\delta)}$ is decreasing for log-concave densities and $\frac{1}{F(\delta)-p}$ is decreasing. For $\delta \in (\underline{\delta}, F^{-1}(p))$,

$$\frac{d}{d\delta}\log(p-F(\delta)) = \frac{f(\delta)}{F(\delta)-p} = -\frac{f(\delta)}{1-F(\delta)}\left(1+\frac{1-p}{p-F(\delta)}\right),$$

is decreasing since $\frac{f(\delta)}{1-F(\delta)}$ and $1+\frac{1-p}{p-F(\delta)}$ are increasing. ■

Lemma 2.

Proof. We only provide the proof for B, as the proof for A is symmetric. Taking the derivative of the monopolistic profit yields

$$\hat{\Pi}_b'(b) = (P-b)\phi_1(b) - \Phi_1(b) = \Phi_1(b)\left((P-b)\frac{\phi_1(b)}{\Phi_1(b)} - 1\right).$$

Proposition 1 implies that $\underline{r}_b$ maximizes $\hat{\Pi}_b$, and thus

$$\hat{\Pi}_b'(\underline{r}_b) = 0 \;\Rightarrow\; (P-\underline{r}_b)\frac{\phi_1(\underline{r}_b)}{\Phi_1(\underline{r}_b)} - 1 = 0. \tag{31}$$

Lemma 15 implies that the function $(P-b)\frac{\phi_1(b)}{\Phi_1(b)}$ is strictly decreasing. It follows that $\hat{\Pi}_b'(b) < 0$ for all $b \in (\underline{r}_b, P)$, and thus $\hat{\Pi}_b^*(b) = \hat{\Pi}_b(b)$ on this interval. Since h is strictly increasing,

$$B(b) = h^{-1}\left(\nu_1\frac{\hat{\Pi}_b^*(\underline{r})}{\hat{\Pi}_b^*(b)}\right) = h^{-1}\left(\nu_1\frac{\hat{\Pi}_b(\underline{r}_b)}{\hat{\Pi}_b(b)}\right)$$

is strictly increasing for $b \in (\underline{r}_b, \overline{r}_b)$.

■

Proposition 2.

Proof. We only provide the proof for $B(\cdot \mid P)$, as the proof for $A(\cdot \mid P)$ is symmetric. The monopolistic profit for a given P is

$$\hat{\Pi}_b(b \mid P) = (P - b)\Phi_1(b).$$

Let $P_2 > P_1$. We first observe from (31) that $\underline{r}_b(P_2) > \underline{r}_b(P_1)$. We conclude the proof by showing that $B(b \mid P_1) > B(b \mid P_2)$ for all $b \in (\underline{r}_b(P_2), \underline{r}_b(P_1))$. Since, from lemma 2, $\hat{\Pi}_b(b \mid P)$ is decreasing from $\underline{r}_b(P)$, this condition is equivalent to

$$B(b \mid P_1) > B(b \mid P_2) \Leftrightarrow \frac{\hat{\Pi}_b(\underline{r}_b(P_2) \mid P_2)}{\hat{\Pi}^*_b(b \mid P_2)} < \frac{\hat{\Pi}_b(\underline{r}_b(P_1) \mid P_1)}{\hat{\Pi}^*_b(b \mid P_1)}$$
$$\Leftrightarrow \frac{\hat{\Pi}_b(b \mid P_2)}{\hat{\Pi}_b(b \mid P_1)} > \frac{\hat{\Pi}_b(\underline{r}_b(P_2) \mid P_2)}{\hat{\Pi}_b(\underline{r}_b(P_1) \mid P_1)}.$$

The last inequality follows since

$$\frac{\partial}{\partial b}\left(\frac{\hat{\Pi}_b(b \mid P_2)}{\hat{\Pi}_b(b \mid P_1)}\right) = \frac{\Phi_1(b)^2}{\hat{\Pi}_b(b \mid P_1)^2}(P_2 - P_1) > 0,$$

so that the ratio is strictly increasing in b. Therefore,

$$\frac{\hat{\Pi}_b(b \mid P_2)}{\hat{\Pi}_b(b \mid P_1)} > \frac{\hat{\Pi}_b(\underline{r}_b(P_2) \mid P_2)}{\hat{\Pi}_b(\underline{r}_b(P_2) \mid P_1)} > \frac{\hat{\Pi}_b(\underline{r}_b(P_2) \mid P_2)}{\hat{\Pi}_b(\underline{r}_b(P_1) \mid P_1)}.$$

■

Lemma 16. *The following three statements are equivalent:*

- $\tilde{\Phi}_1(\cdot \mid P)$ *is decreasing (increasing) in P with respect to FOSD,*
- $\frac{\tilde{\phi}_1(z|P)}{\tilde{\Phi}_1(z|P)}$ *is decreasing (increasing) in P for all* $z \in (0, 1)$*,*
- $\tilde{m}_1(r) \equiv (r - \underline{r})\frac{\phi_1(r)}{\Phi_1(r)}$ *is decreasing (increasing) in r.*

The monotonicity is strict whenever one of these conditions holds strictly.

The following three statements are equivalent:

- $\tilde{\Phi}_0(\cdot \mid P)$ *is decreasing (increasing) in P with respect to FOSD,*
- $\frac{\tilde{\phi}_0(z|P)}{1-\tilde{\Phi}_0(z|P)}$ *is increasing (decreasing) in P for all* $z \in (0, 1)$*,*
- $\tilde{m}_0(r) \equiv (\bar{r} - r)\frac{\phi_0(r)}{1-\Phi_0(r)}$ *is increasing (decreasing) in r.*

The monotonicity is strict whenever one of these conditions holds strictly.

Proof. We only provide the proof for $\tilde{\Phi}_1(\cdot \mid P)$, as the proof for $\tilde{\Phi}_0(\cdot \mid P)$ is symmetric. A direct calculation shows that

$$\frac{\tilde{\phi}_1(z \mid P)}{\tilde{\Phi}_1(z \mid P)} = (P - \underline{r})\frac{\phi_1(\underline{r} + z(P - \underline{r}))}{\Phi_1(\underline{r} + z(P - \underline{r}))} = \frac{1}{z}\tilde{m}_1(\underline{r} + z(P - \underline{r})),$$

which establishes the equivalence between the second and third statements. Another calculation shows that

$$\begin{aligned}\frac{\partial}{\partial P}\tilde{\Phi}_1(z \mid P) &= \frac{\partial}{\partial P}\left(\frac{\Phi_1(\underline{r} + z(P - \underline{r})}{\Phi_1(P)}\right) \\ &= \frac{\Phi_1(\underline{r} + z(P - \underline{r}))}{\Phi_1(P)}\left(\frac{\tilde{m}_1(\underline{r} + z(P - \underline{r})) - \tilde{m}_1(P)}{P - \underline{r}}\right),\end{aligned}$$

which establishes the equivalence between the third and the first statements. ∎

Proposition 3.

Proof. Suppose that $\tilde{\Phi}_1(\cdot \mid P)$ is strictly decreasing in P with respect to FOSD. Equation (31) implies that

$$(1 - \underline{z}_b(P))\frac{\tilde{\phi}_1(\underline{z}_b(P) \mid P)}{\tilde{\Phi}_1(\underline{z}_b(P) \mid P)} - 1 = 0.$$

The log-concavity of $\tilde{\Phi}_1(\cdot \mid P)$, together with Lemma 15 and Lemma 16, implies that if $P_2 > P_1$, then $\underline{z}_b(P_2) < \underline{z}_b(P_1)$. We next define the rescaled monopolistic profit

$$\tilde{\Pi}_b(z \mid P) \equiv (1 - z)\tilde{\Phi}_1(z \mid P).$$

An argument similar to the proof of Proposition 2 shows that $\tilde{B}(\cdot \mid P_1)$ dominates $\tilde{B}(\cdot \mid P_2)$ if

$$\frac{\tilde{\Pi}_b(z \mid P_2)}{\tilde{\Pi}_b(z \mid P_1)} < \frac{\tilde{\Pi}_b(\underline{z}_b(P_2) \mid P_2)}{\tilde{\Pi}_b(\underline{z}_b(P_1) \mid P_1)}$$

for all $z \in (\underline{z}_b(P_1), \overline{z}_b(P_2))$. A direct calculation yields

$$\frac{\partial}{\partial z}\left(\frac{\tilde{\Pi}_b(z \mid P_2)}{\tilde{\Pi}_b(z \mid P_1)}\right) = \frac{(1 - z)^2\tilde{\Phi}_1(z \mid P_1)\tilde{\Phi}_1(z \mid P_2)}{\tilde{\Pi}_b(z \mid P_1)^2}\left(\frac{\tilde{\phi}_1(z \mid P_2)}{\tilde{\Phi}_1(z \mid P_2)} - \frac{\tilde{\phi}_1(z \mid P_1)}{\tilde{\Phi}_1(z \mid P_1)}\right) < 0.$$

Therefore,

$$\frac{\tilde{\Pi}_b(z \mid P_2)}{\tilde{\Pi}_b(z \mid P_1)} < \frac{\tilde{\Pi}_b(\underline{z}_b(P_1) \mid P_2)}{\tilde{\Pi}_b(\underline{z}_b(P_1) \mid P_1)} < \frac{\tilde{\Pi}_b(\underline{z}_b(P_2) \mid P_2)}{\tilde{\Pi}_b(\underline{z}_b(P_1) \mid P_1)}.$$

The remaining cases follow by analogous arguments. ■

Lemma 3.

Proof. For a Beta(a, b) distribution, the rescaled reverse hazard rate is

$$\tilde{m}_1(r) = \frac{r\phi(r)}{\Phi(r)} = \frac{r^a(1-r)^{b-1}}{B(r,a,b)}, \tag{32}$$

where $B(r, a, b)$ is the incomplete beta function. Using integration by parts, we obtain

$$r^a(1-r)^{b-1} = aB(r,a,b) - (b-1)B(r,a+1,b-1).$$

Using the identity $B(r, a+1, b-1) = B(r, a, b-1) - B(r, a, b)$, this can be rewritten as

$$r^a(1-r)^{b-1} = (a+b-1)B(r,a,b) - (b-1)B(r,a,b-1).$$

Substituting into (32) yields

$$\tilde{m}_1(r) = (a+b-1) - (b-1)\frac{B(r,a,b-1)}{B(r,a,b)}.$$

Differentiating gives

$$\frac{d}{dr}\left(\frac{B(r,a,b-1)}{B(r,a,b)}\right) = \frac{r^{a-1}(1-r)^{b-2}}{B(r,a,b)^2}\Big(B(r,a,b) - (1-r)B(r,a,b-1)\Big).$$

The term in parentheses can be rewritten as

$$B(r,a,b) - (1-r)B(r,a,b-1) = \int_0^r t^{a-1}(1-t)^{b-2}(r-t)\,dt > 0.$$

It follows that $\tilde{m}_1'(r) < 0$ when $b > 1$, and $\tilde{m}_1'(r) = 0$ when $b = 1$. The result then follows from Lemma 16. The proof for the exponential distribution is similar, using

$$\tilde{m}_1(r) = \frac{\lambda r e^{\lambda r}}{1 - e^{-\lambda r}} = \frac{\lambda r}{e^{\lambda r} - 1}.$$

■

Lemma 17. *If $\Psi' \succ \Psi$, then for all $q \in (0,1)$,*

- $g(q|\Psi') < g(q|\Psi)$,
- $h(q|\Psi') < h(q|\Psi)$,
- $\frac{1}{\nu_1(\Psi')} h(q|\Psi') > \frac{1}{\nu_1(\Psi)} h(q|\Psi)$.

Proof. To simplify notation, we write $\Psi_2 = \Psi'$ and $\Psi_1 = \Psi$. Define

$$d_k \equiv \psi_k^{(2)} - \psi_k^{(1)}.$$

The definition of $\succ$ implies that the ratio $\psi_k^{(2)}/\psi_k^{(1)}$ is increasing, and strictly so for some k. Together with the normalization $\sum_{k\geq 1} \psi_k^{(1)} = \sum_{k\geq 1} \psi_k^{(2)} = 1$, this implies that there exists $K > 1$ such that $d_k < 0$ for all $k < K$ and $d_k \geq 0$ for all $k \geq K$. Moreover,

$$\sum_{k=1}^{K-1} d_k + \sum_{k=K}^{\infty} d_k = \sum_{k=1}^{\infty} d_k = 0 \Rightarrow \sum_{k=K}^{\infty} d_k = -\sum_{k=1}^{K-1} d_k > 0.$$

For $q \in (0,1)$,

$$\begin{aligned} g_2(q) - g_1(q) &= \sum_{k=1}^{\infty} (\psi_k^{(2)} - \psi_k^{(1)}) q^k = \sum_{k=1}^{\infty} d_k q^k = \sum_{k=1}^{K-1} d_k q^k + q^K \sum_{k=K}^{\infty} d_k q^{k-K} \\ &\leq \sum_{k=1}^{K-1} d_k q^k + q^K \sum_{k=K}^{\infty} d_k = \sum_{k=1}^{K-1} d_k (q^k - q^K) < 0, \end{aligned}$$

which proves the first statement.

For the second statement, note that

$$\frac{\nu_k^{(2)}}{\nu_k^{(1)}} = \left(\frac{\sum_{n\geq 1} n\psi_n^{(1)}}{\sum_{n\geq 1} n\psi_n^{(2)}} \right) \frac{k\psi_k^{(2)}}{k\psi_k^{(1)}} = \left(\frac{\sum_{n\geq 1} n\psi_n^{(1)}}{\sum_{n\geq 1} n\psi_n^{(2)}} \right) \frac{\psi_k^{(2)}}{\psi_k^{(1)}}$$

so the ratio $\nu_k^{(2)}/\nu_k^{(1)}$ is increasing in k. Applying the first statement to the distributions $(\nu_k^{(1)})_{k\geq 1}$ and $(\nu_k^{(2)})_{k\geq 1}$ yields $h_2(q) < h_1(q)$.

For the third statement, since the ratio $\nu_k^{(2)}/\nu_k^{(1)}$ is increasing in k, we have

$$\frac{\nu_1^{(2)}}{\nu_1^{(1)}} < \frac{\nu_k^{(2)}}{\nu_k^{(1)}}$$

for all k such that $\nu_k^{(1)} > 0$. Rearranging gives

$$\frac{\nu_k^{(2)}}{\nu_1^{(2)}} \geq \frac{\nu_k^{(1)}}{\nu_1^{(1)}},$$

with strict inequality for some $k > 1$. ■

Proposition 4.

Proof. Proposition 1 shows that $\underline{r}_b(\Psi)$ is the greatest maximizer of $\hat{\Pi}_b$, that is,

$$\underline{r}_b(\Psi) = \sup\left\{b^* \in [\underline{r}, P) : \hat{\Pi}_b(b^*) \geq \hat{\Pi}_b(b) \text{ for all } b \in [\underline{r}, P)\right\}.$$

It also shows that $\bar{r}_b(\Psi)$ is the smallest value satisfying $\nu_1(\Psi)\hat{\Pi}_b^*(\underline{r}) = \hat{\Pi}_b^*(b)$, namely

$$\bar{r}_b(\Psi) = \inf\left\{b \in (\underline{r}, P) : \nu_1(\Psi)\hat{\Pi}_b^*(\underline{r}) \geq \hat{\Pi}_b^*(b)\right\}.$$

These characterizations imply that $\underline{r}_b(\Psi') = \underline{r}_b(\Psi)$ and $\bar{r}_b(\Psi') > \bar{r}_b(\Psi)$, since $\hat{\Pi}_b$ does not depend on Ψ. For all $b \in (\underline{r}_b(\Psi), \bar{r}_b(\Psi))$, the equilibrium bid distributions satisfy

$$\frac{1}{\nu_1(\Psi')} h(B(b \mid \Psi')|\Psi') = \frac{1}{\nu_1(\Psi)} h(B(b \mid \Psi)|\Psi) = \frac{\hat{\Pi}_b^*(\underline{r})}{\hat{\Pi}_b^*(b)}.$$

Lemma 17 and the monotonicity of $h(\cdot|\Psi)$ imply that $B(b \mid \Psi') < B(b \mid \Psi)$. The proof for A is analogous. ■

B Equilibrium existence proof

B.1 Spaces and operators

We define $\mathcal{P}(\mathcal{D})$ the topological space of distributions on $\mathcal{D}$ equipped with the weak convergence. We also define the space of continuous functions $C(\mathcal{D})$ equipped with the supremum norm $\|\cdot\|_\infty$. We define the monopolistic bid profit operator $\hat{\Pi}_b : \mathcal{P}(\mathcal{D}) \times \mathcal{D} \to C(\mathcal{D})$ as

$$\hat{\Pi}_b(B, \delta^*)(\delta) = \int_\delta^{\delta^*} \frac{1}{r + \gamma + \lambda\tilde{g}(1 - B(x))} dx \int_{\underline{\delta}}^{\delta} \frac{\gamma}{\gamma + \lambda\tilde{g}(1 - B(x))} dF(x) \quad (33)$$
$$\equiv \hat{\Pi}_b(\delta|B, \delta^*),$$

which is the equivalent of (19), where R and Φ_1 are expressed as the solutions of the investors problem and the stationarity condition computed in lemmas 4 and 5. This profit function corresponds to (19) only for $\delta \in [\underline{\delta}, \delta^*]$, and only if $A([\underline{\delta}, \delta^*]) = 0$, which is verified in equilibrium. This however does not hurt the results and is simply a matter of definition. Similarly, we define the monopolistic ask profit function $\hat{\Pi}_a : \mathcal{P}(\mathcal{D}) \times \mathcal{D} \to C(\mathcal{D})$ as

$$\begin{aligned}\hat{\Pi}_a(A, \delta^*)(\delta) &= \int_{\delta^*}^{\delta} \frac{1}{r + \gamma + \lambda \tilde{g}(A(x))} dx \int_{\delta}^{\overline{\delta}} \frac{\gamma}{\gamma + \lambda \tilde{g}(A(x))} dF(x) \\ &\equiv \hat{\Pi}_a(\delta | A, \delta^*),\end{aligned} \tag{34}$$

which is the equivalent of (19) for $\delta \in [\delta^*, \overline{\delta}]$. If $\psi_1 = 1$, the monopolistic profit and the profit functions coincide, since $h(q) = 1$.

Finally, we define the supply operator $\mathcal{S} : \mathcal{P}(\mathcal{D})^2 \to (0, 1)$ as

$$\mathcal{S}(B, A) = \int_{\underline{\delta}}^{\overline{\delta}} \frac{\gamma s + \lambda \tilde{g}\left(A(\delta)\right)}{\gamma + \lambda \Big(\tilde{g}\left(1 - B(\delta)\right) + \tilde{g}\left(A(\delta)\right) \Big)} dF(\delta).$$

Note that this operator returns the total supply implied by B and A, only if $\mathcal{S}(B, A) = s$, which is what is wanted for an equilibrium.

B.2 Optimal pricing

Case $\psi_1 = 1$: We define the optimal bid price correspondence $\mathcal{B} : \mathcal{P}(D) \times \mathcal{D} \to 2^{\mathcal{P}(\mathcal{D})}$

$$\begin{aligned}\mathcal{B}(B, \delta^*) &= \left\{ \beta \in \mathcal{P}(\mathcal{D}) : \mathrm{supp}(\beta) = \underset{\delta \in \mathcal{D}}{\mathrm{argmax}}\, \hat{\Pi}_b(\delta | B, \delta^*) \right\} \\ &= \arg \max_{\beta \in \mathcal{P}(\mathcal{D})} \int_{\mathcal{D}} \hat{\Pi}_b(\delta | B, \delta^*) d\beta(\delta).\end{aligned}$$

This correspondence outputs the set of optimal bid distributions if R and Φ_1 are given by B and δ^*. Similarly, we define $\mathcal{A} : \mathcal{P}(D) \times \mathcal{D} \to 2^{\mathcal{P}(\mathcal{D})}$

$$\begin{aligned}\mathcal{A}(A, \delta^*) &= \left\{ \alpha \in \mathcal{P}(\mathcal{D}) : \mathrm{supp}(\alpha) = \underset{\delta \in \mathcal{D}}{\mathrm{argmax}}\, \hat{\Pi}_a(\delta | A, \delta^*) \right\} \\ &= \arg \max_{\alpha \in \mathcal{P}(\mathcal{D})} \int_{\mathcal{D}} \hat{\Pi}_a(\delta | A, \delta^*) d\alpha(\delta),\end{aligned}$$

which outputs the set of optimal ask distributions if R and Φ_0 are given by A and δ^*.

Lemma 18. *$\mathcal{B}$ and $\mathcal{A}$ are upper hemicontinuous. Moreover, for all $(\phi, \delta^*) \in \mathcal{P}(\mathcal{D}) \times \mathcal{D}$, $\mathcal{B}(\phi, \delta^*)$ and $\mathcal{A}(\phi, \delta^*)$ are non-empty, compact and convex.*

Proof. The firs step is to prove that $\hat{\Pi}_b$ is a continuous operator. Let $(B_n, \delta_n^*) \to (B, \delta^*)$. We first observe that

$$\begin{aligned}
&\sup_{\delta \in \mathcal{D}} \left| \int_{\delta}^{\delta_n^*} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B_n(x))} dx - \int_{\delta}^{\delta^*} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B(x))} dx \right| \\
&\leq \sup_{\delta \in \mathcal{D}} \left| \int_{\delta}^{\delta_n^*} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B_n(x))} dx - \int_{\delta}^{\delta^*} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B_n(x))} dx \right| \\
&+ \sup_{\delta \in \mathcal{D}} \left| \int_{\delta}^{\delta^*} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B_n(x))} dx - \int_{\delta}^{\delta^*} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B(x))} dx \right| \\
&= \sup_{\delta \in \mathcal{D}} \left| \int_{\delta^*}^{\delta_n^*} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B_n(x))} dx \right| \\
&+ \sup_{\delta \in \mathcal{D}} \left| \int_{\delta}^{\delta^*} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B_n(x))} - \frac{1}{r+\gamma+\lambda\tilde{g}(1-B(x))} dx \right| \\
&\leq \frac{|\delta_n^* - \delta^*|}{r+\gamma} + \int_{\underline{\delta}}^{\delta^*} \left| \frac{1}{r+\gamma+\lambda\tilde{g}(1-B_n(x))} - \frac{1}{r+\gamma+\lambda\tilde{g}(1-B(x))} \right| dx \to 0,
\end{aligned}$$

where the convergence of the integral follows from the Portmanteau and dominated convergence theorem. Similarly, one can show that

$$\sup_{\delta \in \mathcal{D}} \left| \int_{\underline{\delta}}^{\delta} \frac{\gamma}{\gamma+\lambda\tilde{g}(1-B_n(x))} dF(x) - \int_{\underline{\delta}}^{\delta} \frac{\gamma}{\gamma+\lambda\tilde{g}(1-B(x))} dF(x) \right| \to 0,$$

in particular the continuity of F is crucial for the application of the dominated convergence theorem. $\hat{\Pi}_b$ is therefore continuous.

We next observe that the mapping $(\beta, B, \delta^*) \to \int_{\mathcal{D}} \hat{\Pi}_b(\delta|B, \delta^*) d\beta(\delta)$ is continuous. Indeed, let $(\beta_n, B_n, \delta_n^*) \to (\beta, B, \delta^*)$ and observe that

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathcal{D}} \hat{\Pi}_b(\delta|B_n, \delta_n^*) d\beta_n(\delta) \\
&= \lim_{n\to\infty} \int_{\mathcal{D}} \left(\hat{\Pi}_b(\delta|B_n, \delta_n^*) - \hat{\Pi}_b(\delta|B, \delta^*) \right) d\beta_n(\delta) + \lim_{n\to\infty} \int_{\mathcal{D}} \hat{\Pi}_b(\delta|B, \delta^*) d\beta_n(\delta) \\
&= 0 + \int_{\mathcal{D}} \hat{\Pi}_b(\delta|B, \delta^*) d\beta(\delta),
\end{aligned}$$

where the convergence of the second term follows from the continuity of $\hat{\Pi}_b(\cdot|B, \delta^*)$ and the definition of the weak convergence, and the convergence of the first term follows from

$$\left| \int_{\mathcal{D}} \left(\hat{\Pi}_b(\delta | B_n, \delta_n^*) - \hat{\Pi}_b(\delta | B, \delta^*) \right) d\beta_n(\delta) \right|$$
$$\leq \int_{\mathcal{D}} \left| \hat{\Pi}_b(\delta | B_n, \delta_n^*) - \hat{\Pi}_b(\delta | B, \delta^*) \right| d\beta_n(\delta) \leq \sup_{\delta \in \mathcal{D}} \left| \hat{\Pi}_b(\delta | B_n, \delta_n^*) - \hat{\Pi}_b(\delta | B, \delta^*) \right|.$$

Since Prokhorov theorem implies that $\mathcal{P}(\mathcal{D})$ is a compact set, we use Berge maximum theorem to conclude that $\mathcal{B}$ is upper hemicontinuous with non-empty and compact values. The convexity of the values of $\mathcal{B}$ follows immediately from its definition. The proof is similar for $\mathcal{A}$. ■

Case $\psi_1 \in (0,1)$: We define the running maximum of the profit function operators

$$\hat{\Pi}_b^*(\delta | B, \delta^*) \equiv \sup_{x \in [\delta, \bar{\delta}]} \hat{\Pi}_b(x | B, \delta^*) \quad \text{and} \quad \hat{\Pi}_a^*(\delta | A, \delta^*) \equiv \sup_{x \in [\underline{\delta}, \delta]} \hat{\Pi}_b(x | A, \delta^*),$$

which allows us to define the bid price distribution offered by dealers as an operator $\mathcal{B} : \mathcal{P}(\mathcal{D}) \times \mathcal{D} \to \mathcal{P}(\mathcal{D})$ such that

$$\mathcal{B}(B, \delta^*)(\delta) = \tilde{\mathbf{h}} \left(\hat{\Pi}_b^*(\underline{\delta} | B, \delta^*), \hat{\Pi}_b^*(\delta | B, \delta^*) \right) \mathbf{1}_{\delta^* > \underline{\delta}} + \mathbf{1}_{\delta \geq \underline{\delta}} \mathbf{1}_{\delta^* = \underline{\delta}}.$$

This operator outputs the solution of the RFQ auction for the bid distribution in line with lemma 6, when R and Φ_1 are generated by B and the inter-dealer price is δ^*. For the ask price distribution, the operator $\mathcal{A} : \mathcal{P}(\mathcal{D}) \times \mathcal{D} \to \mathcal{P}(\mathcal{D})$ is

$$\mathcal{A}(A, \delta^*)(\delta) = \left(1 - \tilde{\mathbf{h}} \left(\hat{\Pi}_a^*(\bar{\delta} | A, \delta^*), \hat{\Pi}_a^*(\delta | A, \delta^*) \right) \right) \mathbf{1}_{\delta^* < \bar{\delta}} + \mathbf{1}_{\delta \geq \bar{\delta}} \mathbf{1}_{\delta^* = \bar{\delta}}.$$

Lemma 19. *$\mathcal{B}$ and $\mathcal{A}$ are continuous operators.*

Proof. Let $(B_n, \delta_n^*) \to (B, \delta^*)$. We first assume that $\delta^* > \underline{\delta}$.

$$\sup_{\delta \in \mathcal{D}} \left| \hat{\Pi}_b^*(\delta | B_n, \delta_n^*) - \hat{\Pi}_b^*(\delta | B, \delta^*) \right| \leq \sup_{\delta \in \mathcal{D}} \sup_{x \in [\underline{\delta}, \delta]} \left| \hat{\Pi}_b(x | B_n, \delta_n^*) - \hat{\Pi}_b(x | B, \delta^*) \right| \tag{35}$$
$$= \sup_{x \in \mathcal{D}} \left| \hat{\Pi}_b(x | B_n, \delta_n^*) - \hat{\Pi}_b(x | B, \delta^*) \right| \to 0,$$

as shown in the proof of proposition 18. Since $\hat{\Gamma}^*_b(\underline{\delta}|B,\delta^*) > 0$, (35) and the continuity of $\tilde{\mathbf{h}}$ for $(x,y) \in \mathbf{R}_{++} \times \mathbf{R}$ imply that for all $\delta \in \mathcal{D}$,

$$\lim_{n\to\infty} \mathcal{B}(B_n,\delta^*_n)(\delta) = \mathcal{B}(B,\delta^*)(\delta).$$

We conclude with Portmanteau theorem, that $\mathcal{B}(B_n,\delta^*_n)$ converges weakly to $\mathcal{B}(B,\delta^*)$. If $\delta^* = \underline{\delta}$, the weak continuity follows from the fact that $\mathcal{B}(B_n,\delta^*_n)$ has values in $[\underline{\delta},\delta^*_n]$. ■

For the next section, to combine the results for $\psi_1 = 1$ and $\psi_1 \in (0,1)$, we change $\mathcal{B}$ and $\mathcal{A}$ defined for $\psi_1 \in (0,1)$ to be singleton correspondences. In particular, they are upper hemicontinuous with non-empty, compact and convex values.

B.3 Equilibrium operator and fixed point

We define the update operator $\Delta^* : \mathcal{P}(\mathcal{D})^2 \times \mathcal{D} \to \mathcal{D}$

$$\Delta^*(B,A,\delta^*) = \delta^* + (\mathcal{S}(B,A) - s)^+ (\overline{\delta} - \delta^*) - (s - \mathcal{S}(B,A))^+ (\delta^* - \underline{\delta}).$$

Δ^* is continuous, since the Portmanteau and dominated convergence theorem imply that $\mathcal{S}$ is continuous. We now proceed to the definition of the equilibrium correspondence $\mathcal{E} : \mathcal{P}(\mathcal{D})^2 \times \mathcal{D} \to 2^{\mathcal{P}(\mathcal{D})^2\times\mathcal{D}}$ defined as

$$\mathcal{E}(B,A,\delta^*) = \mathcal{B}(B,\delta^*) \times \mathcal{A}(A,\delta^*) \times \{\Delta^*(B,A,\delta^*)\}.$$

Theorem 1.

Proof. Lemmas 18 and 19 ensure that $\mathcal{E}$ is upper hemicontinuous, with non-empty, compact and convex values. Moreover, $\mathcal{P}(\mathcal{D})$ is a convex subset of the space $\mathcal{M}(\mathbf{R})$ of signed measures equipped with the weak convergence, which is a Hausdorff locally convex vector space. In particular, Prokhorov theorem implies that $\mathcal{P}(\mathcal{D})$ is compact. Therefore, Kakutani theorem states that the correspondence $\mathcal{E}$ admits a fixed point.

We then observe that for all $(B,A,\beta,\alpha,\delta^*) \in \mathcal{P}(\mathcal{D})^4 \times \mathcal{D}$ such that $\beta \in \mathcal{B}(B,\delta^*)$ and $\alpha \in \mathcal{A}(A,\delta^*)$, $\text{supp}(\beta) \subseteq [\underline{\delta},\delta^*]$ and $\text{supp}(\alpha) \subseteq \left[\delta^*,\overline{\delta}\right]$. Therefore if (B,A,δ^*) is a fixed point of $\mathcal{E}$, the support of B and A must be separated by δ^* and the definitions (33) and (34) are valid for that fixed point.

Let (B,A,δ^*) be a fixed point of $\mathcal{E}$. It only remains to show that $\mathcal{S}(B,A) = s$. Assume towards a contradiction that $\mathcal{S}(B,A) > s$. The definition of Δ^* implies that

$\delta^* = \overline{\delta}$, which in turn implies that $\mathcal{A}(A, \overline{\delta}) = \left\{\mathbf{1}_{\delta \geq \overline{\delta}}\right\} = \{A\}$. However,

$$\mathcal{S}(B, \mathbf{1}_{\delta \geq \overline{\delta}}) = \int_{\underline{\delta}}^{\overline{\delta}} \frac{\gamma s}{\gamma + \lambda \tilde{g}\left(1 - B(\delta)\right)} dF(\delta) \leq s.$$

Assume towards a contradiction that $\mathcal{S}(B, A) < s$. The definition of Δ^* implies that $\delta^* = \underline{\delta}$, which in turn implies that $\mathcal{B}(B, \underline{\delta}) = \left\{\mathbf{1}_{\delta \geq \underline{\delta}}\right\} = \{B\}$. However,

$$\mathcal{S}(\mathbf{1}_{\delta \geq \underline{\delta}}, A) = \int_{\underline{\delta}}^{\overline{\delta}} \frac{\gamma s + \lambda \tilde{g}\left(A(\delta)\right)}{\gamma + \lambda \tilde{g}\left(A(\delta)\right)} dF(\delta) \geq s.$$

■

C Equilibrium characterization proofs

C.1 Necessary and sufficient conditions proofs

Lemma 7.

Proof. The proof is different from Hugonnier et al. (2025), since their proof does not work with $\psi \in (0,1)$. Assume towards a contradiction that there exists $\delta_1, \delta_2 \in \operatorname{supp}(B)$ such that $\delta_1 < \delta_2$ and $(\delta_1, \delta_2) \cap \operatorname{supp}(B) = \emptyset$. This implies that $B(\delta) = b \in (0,1)$ for all $\delta \in [\delta_1, \delta_2]$. In equilibrium, the monopolistic profit reads

$$\hat{\Pi}_b(\delta) = \int_{\delta}^{\delta^*} \frac{1}{r + \gamma + \lambda \tilde{g}(1 - B(x))} dx \int_{\underline{\delta}}^{\delta} \frac{\gamma}{\gamma + \lambda \tilde{g}(1 - B(x))} f(x) dx \tag{36}$$

and is continuously differentiable since f and B are continuous. The dealer profit is $\Pi_b(\delta) = \hat{\Pi}_b(\delta) h(B(\delta))$.

In particular, for all $\delta \in (\delta_1, \delta_2)$, (10) implies $\hat{\Pi}_b(\delta_2) h(B(\delta_2)) \geq \hat{\Pi}_b(\delta) h(B(\delta))$, which in turn implies $\hat{\Pi}_b(\delta_2) \geq \hat{\Pi}_b(\delta)$ since $B(\delta) = B(\delta_2) = b$. Moreover, for all $\delta > \delta_2$, $\hat{\Pi}_b(\delta_2) h(B(\delta_2)) \geq \hat{\Pi}_b(\delta) h(B(\delta))$, which implies $\hat{\Pi}_b(\delta_2) \geq \hat{\Pi}_b(\delta)$ since B is increasing. Therefore, δ_2 is local maximum of $\hat{\Pi}_b$ and $\hat{\Pi}_b'(\delta_2) = 0$.

Similarly, we find that $\hat{\Pi}_b(\delta_1) \geq \hat{\Pi}_b(\delta)$ for all $\delta \in (\delta_1, \delta_2)$. If $\psi_1 = 1$, $\hat{\Pi}_b(\delta) = \Pi_b(\delta)$ and we immediately get that $\hat{\Pi}_b(\delta_1) \geq \hat{\Pi}_b(\delta)$ for all $\delta < \delta_1$. Therefore, $\hat{\Pi}_b'(\delta_1) = 0$ for $\psi_1 = 1$. If $\psi \in (0,1)$, lemma 6 ensures that B is absolutely continuous since the absolute continuity of $\hat{\Pi}_b$ ensures the absolute continuity of $\hat{\Pi}_b^*$. In particular, there exists $\epsilon > 0$ such that $(\delta_1, \delta_1 - \epsilon) \subset \operatorname{supp}(B)$. We can show as before that $\hat{\Pi}_b(\delta) \geq$

$\hat{\Pi}_b(\delta_1)$ for all $\delta \in (\delta_1, \delta_1 - \epsilon)$, which implies that $\hat{\Pi}_b$ is locally not increasing in δ_1. Therefore, $\hat{\Pi}_b'(\delta_1) \leq 0$ for $\psi_1 \in (0, 1]$.

For all $\delta \in [\delta_1, \delta_2]$, we rewrite (36) as

$$\hat{\Pi}_b(\delta) = \frac{\gamma}{\gamma + \lambda \tilde{g}(1-b)} \left(M + \frac{\delta_2 - \delta}{r + \gamma + \lambda \tilde{g}(1-b)} \right) (F(\delta) - p)$$

where

$$M \equiv \int_{\delta_2}^{\delta^*} \frac{1}{r + \gamma + \lambda \tilde{g}(1 - B(x))} dx$$
$$p \equiv F(\delta_1) - \int_{\underline{\delta}}^{\delta_1} \frac{\gamma + \lambda \tilde{g}(1-b)}{\gamma + \lambda \tilde{g}(1 - B(x))} \in (0, F(\delta_1)).$$

Taking the derivative and dividing by $F(\delta) - p$ yields

$$\frac{\hat{\Pi}_b'(\delta)}{F(\delta) - p} = \frac{\gamma}{\gamma + \lambda \tilde{g}(1-b)} \left(\left(M + \frac{\delta_2 - \delta}{r + \gamma + \lambda \tilde{g}(1-b)} \right) \frac{f(\delta)}{F(\delta) - p} - \frac{1}{r + \gamma + \lambda \tilde{g}(1-b)} \right)$$

and lemma 15 implies that it is strictly decreasing over $\delta \in [\delta_1, \delta_2]$. This contradicts $\hat{\Pi}_b'(\delta_1) \geq 0 = \hat{\Pi}_b(\delta_2)$. ■

Lemma 8.

Proof. For all $\delta \in [\underline{\delta}, \underline{\delta}_b)$, (10) implies $\Pi_b(\underline{\delta}_b) = \hat{\Pi}_b(\underline{\delta}_b) \geq \Pi_b(\delta) = \hat{\Pi}_b(\delta)$, since $B(\delta) = B(\underline{\delta}_b) = 0$. Moreover, for all $\delta > \underline{\delta}_b$, $\Pi_b(\underline{\delta}_b) = \hat{\Pi}_b(\underline{\delta}_b) \geq \hat{\Pi}_b(\delta) h(B(\delta)) > \hat{\Pi}_b(\delta)$. Therefore, $\underline{\delta}_b$ is local maximum of $\hat{\Pi}_b$ and $\hat{\Pi}_b'(\underline{\delta}_b) = 0 = \hat{\Pi}_b'(\underline{\delta}_b^-)$. For $\delta \in [\underline{\delta}, \underline{\delta}_b]$, the monopolistic profit (36) reads

$$\hat{\Pi}_b(\delta) = \left(\frac{\underline{\delta}_b - \delta}{r + \gamma + \lambda} + \int_{\underline{\delta}_b}^{\delta^*} \frac{1}{r + \gamma + \lambda \tilde{g}(1 - B(x))} dx \right) \frac{\gamma}{\gamma + \lambda} F(\delta). \tag{37}$$

Furthermore, we observe that since $\underline{\delta}_b \in \text{supp}(B)$ and $h(0) = \nu_1$,

$$\Pi_b(\underline{\delta}_b) = \pi_b^* = \frac{\gamma}{\gamma + \lambda} \int_{\underline{\delta}_b}^{\delta^*} \frac{1}{r + \gamma + \lambda \tilde{g}(1 - B(x))} dx F(\underline{\delta}_b) \nu_1$$

and we obtain that

$$\int_{\underline{\delta}_b}^{\delta^*} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B(x))}dx = \frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)}. \tag{38}$$

Plugging (38) in (37), taking the derivative at $\underline{\delta}_b$ and setting it to 0 yields (20).

For the last assertion, we observe that lemma 15 implies that $\frac{F(\delta)}{f\delta)}F(\delta)$ is strictly increasing on $\delta \in (\underline{\delta}, \overline{\delta})$. In particular, since $\frac{F(\delta)}{f\delta)}$ is increasing, $\lim_{\delta\to\underline{\delta}} \frac{F(\delta)}{f\delta)} < \infty$ and $\lim_{\delta\to\underline{\delta}} \frac{F(\delta)}{f\delta)}F(\delta) = 0$. On the other hand, $\lim_{\delta\to\overline{\delta}} \frac{F(\delta)}{f\delta)}F(\delta) = \lim_{\delta\to\overline{\delta}} \frac{1}{f\delta)} = \frac{1}{f(\underline{\delta})} \in (0,\infty]$. The proof for $\overline{\delta}_a$ is identical. ■

Lemma 9.

Proof. Since $\text{supp}(B) = [\underline{\delta}_b, \overline{\delta}_b]$, (10) and the definition of π_b^* imply that for all $\delta \in [\underline{\delta}_b, \overline{\delta}_b]$

$$\pi_b^* = \int_{\delta}^{\delta^*} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B(x))}dx \int_{\underline{\delta}}^{\delta} \frac{\gamma}{\gamma+\lambda\tilde{g}(1-B(x))}f(x)dxh(B(\delta)).$$

Using (38) and $B(x) = 0$ for all $x \in (\underline{\delta}, \overline{\delta}_b)$, we rewrite the equality as in (21). The proof for the second equality is similar. ■

Lemma 10.

Proof. Since $\overline{\delta}_b \in \text{supp}(B)$ and $h(1) = 1$,

$$\Pi_b(\overline{\delta}_b) = \pi_b^* = \frac{\delta^* - \overline{\delta}_b}{r+\gamma} \int_{\underline{\delta}}^{\overline{\delta}_b} \frac{\gamma f(x)}{\gamma+\lambda\tilde{g}(1-B(x))}dx$$

which yields the first equality. The proof for the second equality is similar. ■

Proposition 6.

Proof. The necessary conditions simply follow from lemmas 8, 10 and 9. Therefore, we only prove that these conditions are also sufficient to characterize an equilibrium. Let B and A that verify the conditions. The corresponding bid profit function is

$$\Pi_b(\delta) = \int_{\delta}^{\delta^*} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B(x))}dx \int_{\underline{\delta}}^{\delta} \frac{\gamma}{\gamma+\lambda\tilde{g}(1-B(x))}f(x)dxh(B(\delta)),$$

where δ^* is defined using (23). This implies that

$$\int_{\underline{\delta}_b}^{\delta^*} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B(x))}dx = \int_{\underline{\delta}_b}^{\overline{\delta}_b} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B(x))}dx + \frac{\delta^*-\overline{\delta}_b}{r+\gamma}$$
$$= \int_{\underline{\delta}_b}^{\overline{\delta}_b} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B(x))}dx + \frac{\pi_b^*}{\int_{\underline{\delta}}^{\overline{\delta}_b} \frac{\gamma f(x)}{\gamma+\lambda\tilde{g}(1-B(x))}dx}$$
$$= \int_{\underline{\delta}_b}^{\overline{\delta}_b} \frac{dx}{r+\gamma+\lambda\tilde{g}(1-B(x))} + \frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\overline{\delta}_b} \frac{dx}{r+\gamma+\lambda\tilde{g}(1-B(x))}$$

where the last equality follows from (21) being verified for $\delta = \overline{\delta}_b$. We therefore rewrite the profit

$$\Pi_b(\delta) = \left(\frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\delta} \frac{1}{r+\gamma+\lambda\tilde{g}(1-B(x))}dx\right)$$
$$\cdot \left(\frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \int_{\underline{\delta}_b}^{\delta} \frac{\gamma}{\gamma+\lambda\tilde{g}(1-B(x))}f(x)dx\right) h(B(\delta)),$$

In particular, since (21) is verified, $\Pi_b(\delta) = \pi_b^*$ for all $\delta \in \left[\underline{\delta}_b, \overline{\delta}_b\right]$.

The monopolistic profit $\hat{\Pi}_b(\delta) = \Pi_b(\delta)/h(B(\delta))$ is continuously differentiable and a small calculation shows that $\hat{\Pi}_b'(\underline{\delta}_b) = \hat{\Pi}_b'(\underline{\delta}_b^-) = 0$, since (20) is verified. For all $\delta < \underline{\delta}_b$, the derivative of $\hat{\Pi}_b(\delta)$ is

$$\hat{\Pi}_b'(\delta) = \frac{\gamma}{\gamma+\lambda}\left(\left(\frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} + \frac{\underline{\delta}_b-\delta}{r+\gamma}\right)f(\delta) - \frac{1}{r+\gamma+\lambda}F(\delta)\right),$$

which implies by lemma 15 that $\frac{\hat{\Pi}_b'(\delta)}{F(\delta)}$ is strictly decreasing on $\delta \in (\underline{\delta}, \underline{\delta}_b)$. Therefore, $\Pi_b'(\delta) = \nu_1\hat{\Pi}_b'(\delta) > 0$ for all $\delta \in (\underline{\delta}, \underline{\delta}_b)$, since $\hat{\Pi}_b'(\underline{\delta}_b) = 0$.

Since $\Pi_b(\delta) = \pi_b^*$ for all $\delta \in \left[\underline{\delta}_b, \overline{\delta}_b\right]$ and $h(B(\delta))$ is increasing in δ, $\hat{\Pi}_b(\delta)$ is decreasing and $\hat{\Pi}_b'(\overline{\delta}_b) \leq 0$. For all $\delta \in (\overline{\delta}_b, \delta^*)$, the derivative of $\hat{\Pi}_b(\delta)$ is

$$\hat{\Pi}_b'(\delta) = \frac{1}{r+\gamma}\left(f(\delta)(\delta^*-\delta) - (F(\delta)-p)\right)$$

where $p = F(\overline{\delta}_b) - \int_{\underline{\delta}}^{\overline{\delta}_b} \frac{\gamma}{\gamma+\lambda\tilde{g}(1-B(x))}f(x)dx \in (0, F(\underline{\delta}_b))$. Therefore $\frac{\hat{\Pi}_b'(\delta)}{F(\delta)-p}$ is strictly decreasing on $\delta \in (\overline{\delta}_b, \delta^*)$ and $\Pi_b'(\delta) = \hat{\Pi}_b'(\delta) < 0$ since $\hat{\Pi}_b'(\overline{\delta}_b) \leq 0$.

$\Pi_b'(\delta) > 0$ for all $\delta \in (\underline{\delta}, \underline{\delta}_b)$, $\Pi_b'(\delta) = 0$ for all $\delta \in (\underline{\delta}_b, \overline{\delta}_b)$ and $\Pi_b'(\delta) < 0$ for all $\delta \in (\overline{\delta}_b, \delta^*)$. It is then immediate that $\text{supp}(B) = \left[\underline{\delta}_b, \overline{\delta}_b\right] = \text{argmax}_{\delta\in[\underline{\delta},\delta^*]}\,\Pi_b(\delta)$.

We can similarly prove that $\text{supp}(A) = [\underline{\delta}_a, \overline{\delta}_a] = \text{argmax}_{\delta \in [\delta^*, \overline{\delta}]} \Pi_a(\delta)$. The proof is complete since the supply equation is verified. ■

C.2 Uniqueness proofs

Lemma 11.

Proof. We only provide the proof for B, since the proof for A is similar. We separate the cases $\psi_1 \in (0,1)$ and $\psi_1 = 1$. We set $\underline{\delta}_b = \underline{\delta}_b(\pi_b^*)$ to alleviate notations.

Case $\psi_1 \in (0,1)$: We define the operator $T : C([\underline{\delta}_b, \overline{\delta}], [0,1]) \to C([\underline{\delta}_b, \overline{\delta}], [0,1])$

$$
\begin{aligned}
T(B)(\delta) &= \tilde{\mathbf{h}}\left(\frac{\pi_b^*}{\nu_1}, \hat{\Pi}_b(\delta)\right) \\
&= \tilde{\mathbf{h}}\left\{\frac{\pi_b^*}{\nu_1}, \left(\frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\delta} \tilde{r}(B(x))dx\right)\left(\frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \int_{\underline{\delta}_b}^{\delta} \tilde{\phi}(B(x))f(x)dx\right)\right\}
\end{aligned}
$$

where $\tilde{\mathbf{h}}(x,y)$ is defined in (6), $\tilde{r}(q) = \frac{1}{r+\gamma+\lambda\tilde{g}(1-q)}$ and $\tilde{\phi}(q) = \frac{\gamma}{\gamma+\lambda\tilde{g}(1-q)}$. $\tilde{r}$ and $\tilde{\phi}$ are Lipschitz with constant K_r and K_ϕ since $g'(1) = \sum_{k=1}^{\infty} \psi_k k < \infty$. Moreover, $\tilde{\mathbf{h}}(\pi_b^*/\nu_1, \cdot)$ is Liptschitz with constant K_h since $\nu_2 > 0$. We also define the restricted operator $T_{\hat{\delta},\epsilon,B^*} : C([\hat{\delta}, \min\{\hat{\delta}+\epsilon, \overline{\delta}\}], [0,1]) \to C([\hat{\delta}, \min\{\hat{\delta}+\epsilon, \overline{\delta}\}], [0,1])$

$$
\begin{aligned}
&T_{\hat{\delta},\epsilon,B^*}(B)(\delta) = \tilde{\mathbf{h}}\Bigg\{\pi_b^*/\nu_1, \\
&\left(\frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\hat{\delta}} \tilde{r}(B^*(x))dx - \int_{\hat{\delta}}^{\delta} \tilde{r}(B(x))dx\right) \\
&\cdot \left(\frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \int_{\underline{\delta}_b}^{\hat{\delta}} \tilde{\phi}(B^*(x))f(x)dx + \int_{\hat{\delta}}^{\delta} \tilde{\phi}(B(x))f(x)dx\right)\Bigg\}
\end{aligned}
$$

where $\hat{\delta} \in [\underline{\delta}_b, \overline{\delta})$, $\epsilon > 0$ and $B^* \in C([\underline{\delta}_b, \hat{\delta}], [0,1])$. Let

$$
\begin{aligned}
R(B)(\delta) &= \frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\hat{\delta}} \tilde{r}(B^*(x))dx - \int_{\hat{\delta}}^{\delta} \tilde{r}(B(x))dx \\
\Phi(B)(\delta) &= \frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \int_{\underline{\delta}_b}^{\hat{\delta}} \tilde{\phi}(B^*(x))f(x)dx + \int_{\hat{\delta}}^{\delta} \tilde{\phi}(B(x))f(x)dx,
\end{aligned}
$$

and observe that $T_{\hat{\delta},\epsilon,B^*}(B)(\delta) = \tilde{\mathbf{h}}\left(\pi_b^*/\nu_1, R(B)(\delta)\Phi(B)(\delta)\right)$.

For all $B_1, B_2 \in C(\left[\hat{\delta}, \min\{\hat{\delta}+\epsilon, \overline{\delta}\}\right], [0,1])$, we observe that

$$\begin{aligned}
&R(B_2)(\delta)\Phi(B_2)(\delta) - R(B_1)(\delta)\Phi(B_1)(\delta) \\
&= R(B_2)(\delta)\left(\Phi(B_2)(\delta) - \Phi(B_1)(\delta)\right) + \Phi(B_1)(\delta)\left(R(B_2)(\delta) - R(B_1)(\delta)\right)
\end{aligned}$$

and thus

$$\begin{aligned}
&|R(B_2)(\delta)\Phi(B_2)(\delta) - R(B_1)(\delta)\Phi(B_1)(\delta)| \\
&\leq R(B_2)(\delta)\left|\Phi(B_2)(\delta) - \Phi(B_1)(\delta)\right| + \Phi(B_1)\left|R(B_2)(\delta) - R(B_1)(\delta)\right| \\
&\leq \left(\frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} + \frac{\overline{\delta}-\underline{\delta}_b}{r+\gamma}\right)\left|\Phi(B_2)(\delta) - \Phi(B_1)(\delta)\right| \\
&+ \left(\frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \overline{\delta} - \underline{\delta}_b\right)\left|R(B_2)(\delta) - R(B_1)(\delta)\right| \\
&\leq K_p\left(K_\phi(F(\delta) - F(\hat{\delta}))\|B_2 - B_1\|_\infty + (\delta - \hat{\delta})K_r\|B_2 - B_1\|_\infty\right) \\
&\leq K_\pi(\delta - \hat{\delta})\|B_2 - B_1\|_\infty,
\end{aligned}$$

where $K_p = \max\left\{\frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} + \frac{\overline{\delta}-\underline{\delta}_b}{r+\gamma}, \frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \overline{\delta} - \underline{\delta}_b\right\}$, $K_\pi = K_p(K_\phi\overline{f} + K_r)$ and $\overline{f}$ is the bound on f implied by lemma 15. Since $\tilde{\mathbf{h}}(\pi_b^*, \cdot)$ is Lipschitz,

$$\begin{aligned}
&\left|T_{\hat{\delta},\epsilon,B^*}(B_2)(\delta) - T_{\hat{\delta},\epsilon,B^*}(B_2)(\delta)\right| \\
=&\left|\tilde{\mathbf{h}}\left(\frac{\pi_b^*}{\nu_1}, R(B_2)(\delta)\Phi(B_2)(\delta)\right) - \tilde{\mathbf{h}}\left(\frac{\pi_b^*}{\nu_1}, R(B_1)(\delta)\Phi(B_1)(\delta)\right)\right| \\
\leq& K_h\left|R(B_2)(\delta)\Phi(B_2)(\delta) - R(B_1)(\delta)\Phi(B_1)(\delta)\right| \\
\leq& K_h K_\pi(\delta - \hat{\delta})\|B_2 - B_1\|_\infty,
\end{aligned}$$

which implies that

$$\left\|T_{\hat{\delta},\epsilon,B^*}(B_2) - T_{\hat{\delta},\epsilon,B^*}(B_2)\right\|_\infty \leq K_T\epsilon\|B_2 - B_1\|_\infty,$$

where $K_T = K_h K_\pi$ is independent of $\hat{\delta}$, ϵ and B^*. In particular, for ϵ small enough, $T_{\hat{\delta},\epsilon,B^*}$ is a contraction and admits a unique fixed point. Using similar arguments as in the standard proof of the Picard–Lindelöf theorem, we conclude that T admits a unique fixed point.

Let B be the fixed point of T. Using lemma 8 and the definition of $\tilde{\mathbf{h}}$, we observe that $B(\underline{\delta}_b) = T(B)(\underline{\delta}_b) = 0$. We define $\overline{\delta}_b = \inf\{\delta \in \left(\underline{\delta}_b, \overline{\delta}\right] : B(\delta) = 1\} \cup \{\overline{\delta}\}$, which is the first δ for which $B(\delta)$ hits 1 (if it never does, it is set to $\overline{\delta}$). The continuity of B implies that $\overline{\delta}_b > \underline{\delta}_b$. If we show that B is strictly increasing in $\left[\underline{\delta}_b, \overline{\delta}_b\right]$, the proof is over since $\tilde{\mathbf{h}}(\pi_b^*/\nu_1, y) = h^{-1}\left(\pi_b^*/y\right) \in (0,1)$ if and only if $\pi_b^*/y \in (\nu_1, 1)$.

To do so, we first show that there exists no $\epsilon > 0$ such that $B(\delta) = 0$ for all $\delta \in \left[\underline{\delta}_b, \underline{\delta}_b + \epsilon\right]$. Indeed, assume that such ϵ exists. For all $\delta \in \left(\underline{\delta}_b, \underline{\delta}_b + \epsilon\right)$

$$\hat{\Pi}_b'(\delta) = \frac{\gamma}{\gamma+\lambda}\left(\left(\frac{\pi_b^*}{\nu_1}\frac{\gamma}{\gamma+\lambda}\frac{1}{F(\underline{\delta}_b)} - \frac{\delta - \underline{\delta}_b}{r+\lambda}\right)f(\delta) - \frac{F(\delta)}{r+\gamma}\right)$$

and we can always pick $\epsilon > 0$ sufficiently small so that $\frac{\pi_b^*}{\nu_1}\frac{\gamma}{\gamma+\lambda}\frac{1}{F(\underline{\delta}_b)} - \frac{\delta-\underline{\delta}_b}{r+\lambda} > 0$. In particular, we observe that $\hat{\Pi}_b'(\underline{\delta}_b) = 0$ and $\frac{\hat{\Pi}_b'(\delta)}{F(\delta)}$ is strictly decreasing. This implies that $\hat{\Pi}_b(\delta) < \hat{\Pi}_b(\underline{\delta}_b) = \pi_b^*$ and $B(\delta) = \tilde{\mathbf{h}}(\pi_b^*, \hat{\Pi}_b(\delta)) > 0$, which is a contradiction.

We now assume towards a contradiction that B is not strictly increasing. There exists $\delta_2 > \delta_1$ such that $B(\delta_2) \leq B(\delta_1)$. We pick $\hat{\delta} \in \operatorname{argmax}_{x\in[\underline{\delta}_b,\delta_2]} B(\delta)\backslash\{\delta_2\} \in (\underline{\delta}_b, \delta_2)$ and $\epsilon < \delta_2 - \hat{\delta}$. It is clear that $\hat{\delta}$ is a local maximum of B, $B(\hat{\delta}) \in (0,1)$ and $F(\hat{\delta}) - \frac{1}{\tilde{\phi}(B(\hat{\delta}))}\left(\frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \int_{\underline{\delta}_b}^{\hat{\delta}}\tilde{\phi}(B(x))f(x)dx\right) > 0$. In particular, $\pi_b^*/\hat{\Pi}_b(\hat{\delta}) \in (1, \frac{1}{\nu_1})$ and $\hat{\delta}$ is a local minimum of $\hat{\Pi}_b$, which implies $\hat{\Pi}_b'(\hat{\delta}) = 0$. For all $\delta \in (\hat{\delta}, \hat{\delta} + \epsilon)$

$$\begin{aligned}
\hat{\Pi}_b'(\hat{\delta}) &= -\tilde{r}(B(\delta))\left(\frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \int_{\underline{\delta}_b}^{\hat{\delta}}\tilde{\phi}(B(x))f(x)dx + \int_{\hat{\delta}}^{\delta}\tilde{\phi}(B(x))f(x)dx\right) \\
&\quad + \tilde{\phi}(B(\delta))f(\delta)\left(\frac{\pi_b^*}{\nu_1}\frac{\gamma}{\gamma+\lambda}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\hat{\delta}}\tilde{r}(B(x))dx - \int_{\hat{\delta}}^{\delta}\tilde{r}(B(x))dx\right) \\
&\leq -\tilde{r}(B(\delta))\left(\frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \int_{\underline{\delta}_b}^{\hat{\delta}}\tilde{\phi}(B(x))f(x)dx + (F(\delta) - F(\hat{\delta}))\tilde{\phi}(b(\epsilon))\right) \\
&\quad + \tilde{\phi}(B(\delta))f(\delta)\left(\frac{\pi_b^*}{\nu_1}\frac{\gamma}{\gamma+\lambda}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\hat{\delta}}\tilde{r}(B(x))dx - \tilde{r}(b(\epsilon))(\delta - \hat{\delta})\right) \\
&= \tilde{\phi}(B(\delta))f(\delta)\left(\frac{\pi_b^*}{\nu_1}\frac{\gamma}{\gamma+\lambda}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\hat{\delta}}\tilde{r}(B(x))dx - \tilde{r}(b(\epsilon))(\delta - \hat{\delta})\right) \\
&\quad - \tilde{r}(B(\delta))\tilde{\phi}(b(\epsilon))(F(\delta) - p(\epsilon)) \equiv M(\delta),
\end{aligned}$$

where $p(\epsilon) = F(\hat{\delta}) - \frac{1}{\tilde{\phi}(b(\epsilon))}\left(\frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \int_{\underline{\delta}_b}^{\hat{\delta}}\tilde{\phi}(B(x))f(x)dx\right) < F(\hat{\delta})$ and $b(\epsilon) = \min_{x\in\left[\hat{\delta},\hat{\delta}+\epsilon\right]} B(x)$. We pick ϵ small enough so that $p(\epsilon) > 0$ and

$$\frac{\pi_b^*}{\nu_1}\frac{\gamma}{\gamma+\lambda}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\hat{\delta}}\tilde{r}(B(x))dx - \tilde{r}(b(\epsilon))\epsilon > 0.$$

In particular, $M(\hat{\delta}) = \hat{\Pi}'_b(\hat{\delta}) = 0$ and

$$\frac{M(\delta)}{(F(\delta)-p(\epsilon))\tilde{\phi}(B(\delta))} = -\tilde{\phi}(b(\epsilon))\frac{\tilde{r}(B(\delta))}{\tilde{\phi}(B(\delta))}$$
$$\frac{f(\delta)}{F(\delta)-p(\epsilon)}\left(\frac{\pi_b^*}{\nu_1}\frac{\gamma}{\gamma+\lambda}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\hat{\delta}} \tilde{r}(B(x))dx - \tilde{r}(b(\epsilon))(\delta-\hat{\delta})\right).$$

Since $\frac{\tilde{r}(B(\hat{\delta}))}{\tilde{\phi}(B(\hat{\delta}))} \leq \frac{\tilde{r}(B(\delta))}{\tilde{\phi}(B(\delta))}$ and $\frac{f(\delta)}{F(\delta)-p(\epsilon)}$ is decreasing, $\frac{M(\delta)}{(F(\delta)-p(\epsilon))\tilde{\phi}(B(\delta))} < \frac{M(\hat{\delta})}{(F(\hat{\delta})-p(\epsilon))\tilde{\phi}(B(\hat{\delta}))} = 0$ and we conclude that $0 > M(\delta) \geq \hat{\Pi}'_b(\delta)$, which contradicts $\hat{\delta}$ being a local minimum.

Case $\psi_1 = 1$: If a solution B exists, we can find a function β such that

$$\beta(\delta) = \frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \int_{\underline{\delta}_b}^{\delta} \frac{\gamma}{\gamma+\lambda(1-B(x))}f(x)dx. \tag{39}$$

β uniquely determines B since

$$B(\delta) = 1 - \frac{\gamma}{\lambda}\left(\frac{f(\delta)}{\beta'(\delta)} - 1\right). \tag{40}$$

We observe that $\beta(\underline{\delta}_b) = \frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b)$. (39) and (40) imply that the functional equation (21) can be written for all $\delta \in [\underline{\delta}_b, \overline{\delta}_b]$

$$\pi_b^* = \left(\frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\delta}\frac{\beta'(x)}{r\beta'(x)+\gamma f(x)}dx\right)\beta(\delta).$$

Taking the derivative on both sides yields the following Riccati differential equation

$$\begin{cases} r\beta'(\delta) = \frac{1}{\pi_b^*}\beta(\delta)^2 - \gamma f(\delta) \\ \beta(\underline{\delta}_b) = \frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b). \end{cases}$$

which is valid for all $\delta \in (\underline{\delta}_b, \overline{\delta}_b)$. Since $B(\delta) \in [0,1]$, (40) implies that

$$\frac{\gamma}{\gamma+\lambda}f(\delta) \leq \beta'(\delta) \leq f(\delta).$$

We therefore restrict the differential equation to

$$\begin{cases} r\beta'(\delta) = \min\{\max\{\frac{\gamma}{\gamma+\lambda} rf(\delta), \frac{1}{\pi_b^*}\beta(\delta)^2 - \gamma f(\delta)\}, rf(\delta)\} \\ \beta(\underline{\delta}_b) = \frac{\gamma}{\gamma+\lambda} F(\underline{\delta}_b). \end{cases} \tag{41}$$

We now show that we can invert the process to find a unique B. The Picard–Lindelöf theorem implies that (41) admits a unique solution that we denote β, which is continuously differentiable. Using (20), a straightforward calculation shows that

$$\beta'(\underline{\delta}_b) = \frac{\gamma}{\gamma+\lambda} f(\underline{\delta}_b).$$

In particular, there exists no $\epsilon > 0$ such that $\beta'(\delta) = \frac{\gamma}{\gamma+\lambda} f(\delta)$ for all $\delta \in [\underline{\delta}_b, \underline{\delta}_b + \epsilon]$. Indeed, if there is, $\beta(\delta) = \frac{\gamma}{\gamma+\lambda} F(\delta)$. However lemma 15 implies that $\frac{1}{\pi_b^*}\left(\frac{\gamma}{\gamma+\lambda}\right)^2 \frac{F(\delta)^2}{f(\delta)} - \gamma > \frac{1}{\pi_b^*}\left(\frac{\gamma}{\gamma+\lambda}\right)^2 \frac{F(\underline{\delta}_b)^2}{f(\underline{\delta}_b)} - \gamma = r\frac{\gamma}{\gamma+\lambda}$, which contradicts (41).

We now define $\overline{\delta}_b = \inf\{\delta \in \left(\underline{\delta}_b, \overline{\delta}\right] : \beta'(\delta) = f(\delta)\} \cup \{\overline{\delta}\}$. $\beta'(\delta)/f(\delta)$ is strictly increasing on $\delta \in \left[\underline{\delta}_b, \overline{\delta}_b\right]$. Indeed, we assume towards a contradiction that it is not the case. The previous paragraph implies that there exists $\hat{\delta} \in \left(\underline{\delta}_b, \overline{\delta}_b\right)$ and $\epsilon > 0$ such that $\beta'(\hat{\delta})/f(\hat{\delta}) \geq \beta'(\delta)/f(\delta)$ for all $\delta \in \left[\hat{\delta}, \hat{\delta} + \epsilon\right]$ and $\beta'(\hat{\delta})/f(\hat{\delta}) > \frac{\gamma}{\gamma+\lambda}$ (the intuition is similar to the case $\psi_1 \in (0,1)$). In particular, for ϵ small enough, (41) implies that $r\beta'(\delta) = \frac{1}{\pi_b^*}\beta(\delta)^2 - \gamma f(\delta)$, which in turn implies that $\beta(\hat{\delta})^2 \geq \beta(\delta)^2$ and $\beta(\hat{\delta}) \geq \beta(\delta)$. However, this contradicts $\beta'(\hat{\delta}) > 0$ as implied by (41).

The strict increase of $\beta'(\delta)/f(\delta)$ allows us to infer that for all $\delta \in \left(\underline{\delta}_b, \overline{\delta}_b\right)$, $r\beta'(\delta) = \frac{1}{\pi_b^*}\beta(\delta)^2 - \gamma f(\delta)$. A small calculation using this and (20) shows that

$$\begin{aligned} &\left(\frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\delta} \frac{\beta'(x)}{r\beta'(x) + \gamma f(x)} dx\right)\beta(\delta) \\ &= \pi_b^*\left(\frac{1}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\delta} \frac{\beta'(x)}{\beta^2(x)} dx\right)\beta(\delta) \\ &= \pi_b^*\left(\frac{1}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} + \frac{1}{\beta(\delta)} - \frac{1}{\beta(\underline{\delta}_b)}\right)\beta(\delta) = \pi_b^*. \end{aligned}$$

We define B following (40) for all $\delta \in \left(\underline{\delta}_b, \overline{\delta}_b\right)$. In particular B solves (21), B is strictly increasing , $B(\underline{\delta}_b) = 0$ and $B(\overline{\delta}_b) = 1$ if $\overline{\delta}_b < \overline{\delta}$. ■

Lemma 12.

Proof. It follows from the strict increase of $\underline{\delta}_b(\cdot)$ and from lemma 11, that $B(\delta|\pi_b^{(2)}) < B(\delta|\pi_b^{(1)})$ on a set of positive measure. We now prove $B(\delta|\pi_b^{(2)}) \leq B(\delta|\pi_b^{(1)})$ for B only, as the proof for A is similar. To alleviate notation, we set $\underline{\delta}_b^{(2)} = \underline{\delta}_b(\pi_b^{(2)})$, $\underline{\delta}_b^{(1)} = \underline{\delta}_b(\pi_b^{(1)})$, $\overline{\delta}_b^{(2)} = \overline{\delta}_b(\pi_b^{(2)})$, $\overline{\delta}_b^{(1)} = \overline{\delta}_b(\pi_b^{(1)})$, $B_2(\delta) = B(\delta|\pi_b^{(2)})$ and $B_1(\delta) = B(\delta|\pi_b^{(1)})$. We separate the cases $\psi_1 \in (0,1)$ and $\psi_1 = 1$.

Case $\psi_1 \in (0,1)$: We define

$$U = \left\{\delta \in \left(\underline{\delta}_b^{(2)}, \overline{\delta}_b^{(1)}\right) : B_1(\delta) = B_2(\delta)\right\}.$$

We assume towards a contradiction that $U \neq \emptyset$ and we define $\hat{\delta} = \inf U$. The continuity of B_1 and B_2 implies that $B_1(\hat{\delta}) = B_2(\hat{\delta}) = b \in (0,1)$. In particular, for all $\delta \in \left(\underline{\delta}_b^{(2)}, \hat{\delta}\right)$, $B_1(\delta) - B_2(\delta) > 0$. Moreover, B_1 and B_2 are differentiable on $\delta \in \left(\underline{\delta}_b^{(2)}, \hat{\delta}\right]$ since

$$B_i(\delta) = h^{-1}\left\{\left(\frac{1}{\pi_b^{(i)}}\left(\frac{\pi_b^{(i)}}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\delta}\frac{1}{r+\gamma+\lambda\tilde{g}(1-B_i(x))}dx\right)\right.\right.$$
$$\left.\left.\cdot\left(\frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \int_{\underline{\delta}_b}^{\delta}\frac{\gamma}{\gamma+\lambda\tilde{g}(1-B_i(x))}f(x)dx\right)\right)^{-1}\right\}.$$

In particular, $B_2'(\hat{\delta}) \geq B_1'(\hat{\delta})$, which implies that $\frac{1}{\pi_b^{(2)}}\hat{\Pi}_2'(\hat{\delta}) \leq \frac{1}{\pi_b^{(1)}}\hat{\Pi}_1'(\hat{\delta})$, where $\hat{\Pi}_i$ is the monopolistic profit associated with $\pi_b^{(i)}$ and B_i.

$$\begin{aligned}\frac{1}{\pi_b^{(i)}}\hat{\Pi}_b'(\hat{\delta}) =& \tilde{\phi}(b)f(\hat{\delta})\left(\frac{1}{\nu_1}\frac{\gamma}{\gamma+\lambda}\frac{1}{F(\underline{\delta}_b^{(i)})} - \frac{1}{\pi_b^{(i)}}\int_{\underline{\delta}_b^{(i)}}^{\hat{\delta}}\tilde{r}(B_i(x))dx\right)\\ &- \tilde{r}(b)\frac{1}{\pi_b^{(i)}}\int_{\underline{\delta}}^{\hat{\delta}}\tilde{\phi}(B_i(x))f(x)dx\end{aligned}$$

and we observe that

$$\frac{1}{\pi_b^{(2)}}\int_{\underline{\delta}}^{\hat{\delta}}\tilde{\phi}(B_2(x))f(x)dx < \frac{1}{\pi_b^{(1)}}\int_{\underline{\delta}}^{\hat{\delta}}\tilde{\phi}(B_1(x))f(x)dx \tag{42}$$

since $B_2(\delta) \leq B_1(\delta)$ for all $\delta < \hat{\delta}$ and $B_2(\delta) \leq B_1(\delta)$ for $\delta \in (\underline{\delta}_b^{(1)}, \underline{\delta}_b^{(1)})$. Moreover

$$\frac{1}{\nu_1}\frac{\gamma}{\gamma+\lambda}\frac{1}{F(\underline{\delta}_b^{(1)})} - \frac{1}{\pi_b^{(1)}}\int_{\underline{\delta}_b^{(1)}}^{\hat{\delta}}\tilde{r}(B_1(x))dx < \ell(\pi_b^{(1)}), \tag{43}$$

where we define

$$\ell(\pi) = \frac{1}{\nu_1}\frac{\gamma}{\gamma+\lambda}\frac{1}{F(\underline{\delta}_b(\pi))} - \frac{1}{\pi}\int_{\underline{\delta}_b(\pi)}^{\hat{\delta}} \tilde{r}(B_2(x))dx$$

for all $\pi \in \left[\pi_b^{(1)}, \pi_b^{(2)}\right]$. In particular, $\underline{\delta}_b(\pi)$ is absolutely continuous since (20) shows that it is defined as the inverse of $\frac{F(\delta)}{f(\delta)}F(\delta)$, which is absolutely continuous with strictly positive derivative. The derivative of ℓ is thus

$$\begin{aligned}\ell'(\pi) &= \left(\frac{1}{\pi}\tilde{r}(B_2(\underline{\delta}_b(\pi))) - \frac{1}{\nu_1}\frac{\gamma}{\gamma+\lambda}\frac{f(\underline{\delta}_b(\pi))}{F(\underline{\delta}_b(\pi))^2}\right)\underline{\delta}_b'(\pi) + \frac{1}{\pi^2}\int_{\underline{\delta}_b(\pi)}^{\hat{\delta}} \tilde{r}(B_2(x))dx \\ &= \left(\frac{1}{\pi}\frac{1}{r+\gamma+\lambda} - \frac{1}{\pi}\frac{1}{r+\gamma+\lambda}\right)\underline{\delta}_b'(\pi) + \frac{1}{\pi^2}\int_{\underline{\delta}_b(\pi)}^{\hat{\delta}} \tilde{r}(B_2(x))dx > 0,\end{aligned}$$

where the last equality from (20) and $\underline{\delta}_b(\pi) \leq \underline{\delta}_b^{(2)}$, and therefore $\ell(\pi_b^{(1)}) < \ell(\pi_b^{(2)})$. We conclude that (42) and (43) imply that $\frac{1}{\pi_b^{(2)}}\hat{\Pi}_2'(\hat{\delta}) > \frac{1}{\pi_b^{(1)}}\hat{\Pi}_1'(\hat{\delta})$, which is a contradiction.

Case $\psi_1 = 1$: We define β_1 and β_2

$$\beta_i(\delta) = \frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b^{(i)}) + \int_{\underline{\delta}_b^{(i)}}^{\delta} \frac{\gamma}{\gamma+\lambda(1-B_i(x))}f(x)dx.$$

and we observe that

$$B_i(\delta) = 1 - \frac{\gamma}{\lambda}\left(\frac{f(\delta)}{\beta_i'(\delta)} - 1\right).$$

The increase of B_i implies the increase of $\frac{\beta_i'(\delta)}{f(\delta)}$. As in the proof of proposition 11, we show that for all $\delta \in (\underline{\delta}_b^{(i)}, \overline{\delta}_b^{(i)})$

$$r\beta_i'(\delta) = \frac{1}{\pi_b^{(i)}}\beta_i(\delta)^2 - \gamma f(\delta). \tag{44}$$

and $\beta_i'(\underline{\delta}_b^{(i)}) = \frac{\gamma}{\gamma+\lambda}f(\underline{\delta}_b^{(i)})$. We define

$$U = \left\{\delta \in \left(\underline{\delta}_b^{(2)}, \overline{\delta}_b^{(1)}\right) : B_1(\delta) = B_2(\delta)\right\}.$$

We assume towards a contradiction that $U \neq \emptyset$ and we define $\hat{\delta} = \inf U$. Therefore, $B_1(\hat{\delta}) = B_2(\hat{\delta}) \in (0,1)$ and $\frac{\beta_1'(\hat{\delta})}{f(\hat{\delta})} = \frac{\beta_2'(\hat{\delta})}{f(\hat{\delta})}$. (44) implies that $\frac{1}{\pi_b^{(1)}}\beta_1(\hat{\delta})^2 = \frac{1}{\pi_b^{(2)}}\beta_2(\hat{\delta})^2$, and thus $\beta_1(\hat{\delta}) < \beta_2(\hat{\delta})$. Moreover, since $\frac{\beta_1'(\underline{\delta}_b^{(2)})}{f(\underline{\delta}_b^{(2)})} > \frac{\beta_2'(\underline{\delta}_b^{(2)})}{f(\underline{\delta}_b^{(2)})}$, $\frac{\beta_1'(\delta)}{f(\delta)} > \frac{\beta_2'(\delta)}{f(\delta)}$ for all $\delta \in (\underline{\delta}_b^{(2)}, \hat{\delta})$. However, we obtain the following contradiction:

$$\begin{aligned}
\beta_1(\hat{\delta}) &= \frac{\gamma}{\gamma+\lambda} F(\underline{\delta}_b^{(1)}) + \int_{\underline{\delta}_b^{(1)}}^{\hat{\delta}} \beta_1'(x)dx \\
&= \frac{\gamma}{\gamma+\lambda} F(\underline{\delta}_b^{(1)}) + \int_{\underline{\delta}_b^{(1)}}^{\underline{\delta}_b^{(2)}} \frac{\beta_1'(x)}{f(x)} f(x)dx + \int_{\underline{\delta}_b^{(2)}}^{\hat{\delta}} \frac{\beta_1'(x)}{f(x)} f(x)dx \\
&> \frac{\gamma}{\gamma+\lambda} F(\underline{\delta}_b^{(1)}) + \int_{\underline{\delta}_b^{(1)}}^{\underline{\delta}_b^{(2)}} \frac{\gamma}{\gamma+\lambda} f(x)dx + \int_{\underline{\delta}_b^{(2)}}^{\hat{\delta}} \frac{\beta_1'(x)}{f(x)} f(x)dx \\
&> \frac{\gamma}{\gamma+\lambda} F(\underline{\delta}_b^{(2)}) + \int_{\underline{\delta}_b^{(2)}}^{\hat{\delta}} \frac{\beta_2'(x)}{f(x)} f(x)dx = \beta_2(\hat{\delta}).
\end{aligned}$$

We now turn to the second assertion. We only prove it for δ_b^*, since the proof for δ_a^* is similar. We assume that $\delta_b^*(\pi_b^{(2)}) < \bar{\delta}$ and $\bar{\delta}_b^{(2)} < \delta_b^*(\pi_b^{(1)})$ otherwise the result is trivial.

$$\begin{aligned}
\frac{\delta_b^*(\pi_b^{(i)}) - \bar{\delta}_b^{(2)}}{r+\gamma} &= \frac{\delta_b^*(\pi_b^{(i)}) - \bar{\delta}_b^{(i)}}{r+\gamma} - \frac{\bar{\delta}_b^{(2)} - \bar{\delta}_b^{(i)}}{r+\gamma} \\
&= \frac{\pi_b^{(i)}}{\int_{\underline{\delta}}^{\bar{\delta}_b^{(i)}} \frac{\gamma f(x)}{\gamma+\lambda \tilde{g}(1-B_i(x))} dx} - \int_{\bar{\delta}_b^{(i)}}^{\bar{\delta}_b^{(2)}} \tilde{r}(B_i(x))dx \\
&= \frac{\pi_b^{(i)}}{\nu_1} \frac{\gamma+\lambda}{\gamma} \frac{1}{F(\underline{\delta}_b^{(i)})} - \int_{\underline{\delta}_b^{(i)}}^{\bar{\delta}_b^{(i)}} \tilde{r}(B_i(x))dx - \int_{\bar{\delta}_b^{(i)}}^{\bar{\delta}_b^{(2)}} \tilde{r}(B_i(x))dx \\
&= \frac{\pi_b^{(i)}}{\nu_1} \frac{\gamma+\lambda}{\gamma} \frac{1}{F(\underline{\delta}_b^{(i)})} - \int_{\underline{\delta}_b^{(i)}}^{\bar{\delta}_b^{(2)}} \tilde{r}(B_i(x))dx
\end{aligned}$$

and the result follows from

$$\frac{\pi_b^{(1)}}{\nu_1} \frac{\gamma+\lambda}{\gamma} \frac{1}{F(\underline{\delta}_b^{(1)})} - \int_{\underline{\delta}_b^{(1)}}^{\bar{\delta}_b^{(2)}} \tilde{r}(B_1(x))dx < \frac{\pi_b^{(2)}}{\nu_1} \frac{\gamma+\lambda}{\gamma} \frac{1}{F(\underline{\delta}_b^{(2)})} - \int_{\underline{\delta}_b^{(2)}}^{\bar{\delta}_b^{(2)}} \tilde{r}(B_2(x))dx,$$

as shown in the proof of the first assertion for $\psi_1 \in (0,1)$.

To prove the third assertion, we define the function

$$\mathcal{S}(m|\pi_b^*, \pi_a^*) = \int_{\underline{\delta}}^{\bar{\delta}} \frac{\gamma m + \lambda \tilde{g}(A(x|\pi_a^*))}{\gamma + \lambda(\tilde{g}(1 - B(x|\pi_b^*)) + \tilde{g}(A(x|\pi_a^*)))} f(x)dx.$$

It is trivial using the first assertion that $\mathcal{S}(m|\pi_b^*, \pi_a^*)$ is strictly increasing in m, strictly decreasing in π_b^*, and strictly increasing in π_a^*. In particular, $s(\pi_b^*, \pi_a^*)$ is the unique fixed point of $\mathcal{S}(m|\pi_b^*, \pi_a^*)$ for π_b^* and π_a^* fixed. We conclude the proof of the lemma by showing that $s(\pi_b^*, \pi_a^*)$ is strictly decreasing in π_b^* and strictly increasing in π_a^*. Indeed, we observe that

$$0 < \frac{\partial \mathcal{S}}{\partial m}(m|\pi_b^*, \pi_a^*) = \int_{\underline{\delta}}^{\overline{\delta}} \frac{\gamma}{\gamma + \lambda(\tilde{g}(1 - B(x|\pi_b^*)) + \tilde{g}(A(x|\pi_a^*)))} f(x)dx < 1,$$

from which we infer

$$\begin{aligned} &m > s(\pi_b^*, \pi_a^*) \Leftrightarrow \mathcal{S}(m|\pi_b^*, \pi_a^*) < m \quad \text{and} \\ &m < s(\pi_b^*, \pi_a^*) \Leftrightarrow \mathcal{S}(m|\pi_b^*, \pi_a^*) > m. \end{aligned}$$

In particular,

$$\mathcal{S}(s(\pi_b^{(2)}, \pi_a^*)|\pi_b^{(1)}, \pi_a^*) > \mathcal{S}(s(\pi_b^{(2)}, \pi_a^*)|\pi_b^{(2)}, \pi_a^*) = s(\pi_b^{(2)}, \pi_a^*),$$

which implies that $s(\pi_b^{(2)}, \pi_a^*) < s(\pi_b^{(1)}, \pi_a^*)$ and

$$\mathcal{S}(s(\pi_b^*, \pi_a^{(2)})|\pi_b^*, \pi_a^{(1)}) < \mathcal{S}(s(\pi_b^*, \pi_a^{(2)})|\pi_b^*, \pi_a^{(2)}) = s(\pi_b^*, \pi_a^{(2)}),$$

which implies that $s(\pi_b^*, \pi_a^{(2)}) > s(\pi_b^*, \pi_a^{(1)})$. ■

C.3 Implementation proofs and details

Proposition 7.

Proof. The result is established as part of the proof of lemma 11. ■

Proposition 8.

Proof. We prove the result for $\hat{B}_N$; the proof for $\hat{A}_N$ is similar. The proof resembles the standard proof of the convergence of the Euler method. To alleviate notation, we set

$B(\delta) = B(\delta|\pi_b^*)$. We define $\hat{R}_n$ and $\hat{\Phi}_n$

$$\hat{R}_n = \frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} - \Delta_b \sum_{i=0}^{n-1} \tilde{r}(\hat{B}_i^{(N)})$$
$$\hat{\Phi}_n = \frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \sum_{i=0}^{n-1} \tilde{\phi}(\hat{B}_i^{(N)}) \left(F(\delta_{i+1}^{(b)}) - F(\delta_i^{(b)}) \right),$$

as well as $R(\delta)$ and $\Phi(\delta)$

$$R(\delta) = \frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} - \int_{\underline{\delta}_b}^{\delta} \tilde{r}(B(x))dx$$
$$\Phi(\delta) = \frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \int_{\underline{\delta}_b}^{\delta} \tilde{\phi}(B(x))f(x)dx.$$

In particular, $\hat{B}_n^{(N)} = \tilde{\mathbf{h}}(\frac{\pi_b^*}{\nu_1}, \hat{R}_n\hat{\Phi}_n)$ and $B(\delta) = \tilde{\mathbf{h}}(\frac{\pi_b^*}{\nu_1}, R(\delta)\Phi(\delta))$. Moreover, we define $\hat{\epsilon}_n = R(\delta_n^{(b)})\Phi(\delta_n^{(b)}) - \hat{R}_n\hat{\Phi}_n$. A small calculation shows that

$$\begin{aligned}
|\hat{\epsilon}_n| &= |R(\delta_n^{(b)})(\Phi(\delta_n^{(b)}) - \hat{\Phi}_n) + \hat{\Phi}_n(R(\delta_n^{(b)}) - \hat{R}_n)| \\
&\leq |R(\delta_n^{(b)})||\Phi(\delta_n^{(b)}) - \hat{\Phi}_n| + |\hat{\Phi}_n||R(\delta_n^{(b)}) - \hat{R}_n| \\
&\leq K_p(|\Phi(\delta_n^{(b)}) - \hat{\Phi}_n| + |R(\delta_n^{(b)}) - \hat{R}_n|)
\end{aligned} \tag{45}$$

where $K_p = \max\left\{ \frac{\pi_b^*}{\nu_1}\frac{\gamma+\lambda}{\gamma}\frac{1}{F(\underline{\delta}_b)} + \frac{\overline{\delta}-\underline{\delta}_b}{r+\gamma}, \frac{\gamma}{\gamma+\lambda}F(\underline{\delta}_b) + \overline{\delta} - \underline{\delta}_b \right\}$. We now turn to:

$$\begin{aligned}
|\Phi(\delta_n^{(b)}) - \hat{\Phi}_n| &\leq \sum_{i=0}^{n-1} \int_{\delta_i^{(b)}}^{\delta_{i+1}^{(b)}} |\tilde{\phi}(B(x)) - \tilde{\phi}(\hat{B}_i^{(N)})| f(x)dx \\
&\leq K_\phi \sum_{i=0}^{n-1} \int_{\delta_i^{(b)}}^{\delta_{i+1}^{(b)}} |B(x) - \hat{B}_i^{(N)}| f(x)dx \\
&= K_\phi \sum_{i=0}^{n-1} \int_{\delta_i^{(b)}}^{\delta_{i+1}^{(b)}} \left| \tilde{\mathbf{h}}(\frac{\pi_b^*}{\nu_1}, R(x)\Phi(x)) - \tilde{\mathbf{h}}(\frac{\pi_b^*}{\nu_1}, \hat{R}_i\hat{\Phi}_i) \right| f(x)dx \\
&\leq K_\phi K_h \sum_{i=0}^{n-1} \int_{\delta_i^{(b)}}^{\delta_{i+1}^{(b)}} |R(x)\Phi(x) - \hat{R}_i\hat{\Phi}_i| f(x)dx \\
&\leq K_\phi K_h \sum_{i=0}^{n-1} \int_{\delta_i^{(b)}}^{\delta_{i+1}^{(b)}} \left(|R(x)\Phi(x) - R(\delta_i^{(b)})\Phi(\delta_i^{(b)})| + |\hat{\epsilon}_i| \right) f(x)dx.
\end{aligned}$$

We define the modulus of continuity $\omega(\Delta) = \max_{|x-y|} |R(x)\Phi(x) - R(y)\Phi(y)|$, which exists since $R(\delta)\Phi(\delta)$ is uniformly continuous. Therefore, the inequality becomes

$$
\begin{aligned}
|\Phi(\delta_n^{(b)}) - \hat{\Phi}_n| &\leq K_\phi K_h \sum_{i=0}^{n-1} \int_{\delta_i^{(b)}}^{\delta_{i+1}^{(b)}} (|\omega(\Delta_b)| + |\hat{\epsilon}_i|) f(x)dx \\
&\leq K_\phi K_h \left((1 - F(\underline{\delta}_b))\omega(\Delta_b) + \sum_{i=0}^{n-1} |\hat{\epsilon}_i| (F(\delta_{i+1}^{(b)}) - F(\delta_i^{(b)})) \right) \\
&\leq K_\phi K_h \left((1 - F(\underline{\delta}_b))\omega(\Delta_b) + \overline{f}\Delta_b \sum_{i=0}^{n-1} |\hat{\epsilon}_i| \right)
\end{aligned}
$$

Similarly, we can show

$$
|\Phi(\delta_n^{(b)}) - \hat{\Phi}_n| \leq K_r K_h \left((\overline{\delta} - \underline{\delta}_b)\omega(\Delta_b) + \Delta_b \sum_{i=0}^{n-1} |\hat{\epsilon}_i| \right).
$$

Plugging this in (45) yields

$$
|\hat{\epsilon}_n| \leq K_1 \omega(\Delta_b) + K_2 \Delta_b \sum_{i=0}^{n-1} |\epsilon_i|,
$$

where $K_1 = K_p K_h (K_r(\overline{\delta} - \underline{\delta}_b) + K_\phi(1 - F(\underline{\delta}_b)))$ and $K_2 = K_p K_h (K_r + \overline{f} K_\phi)$. The discrete Grönwall inequality therefore implies that

$$
|\hat{\epsilon}_n| \leq K_1 \omega(\Delta_b) \exp(K_2 \sum_{i=0}^{n-1} \Delta_b) \leq K_1 \omega(\Delta_b) \exp(K_2(\overline{\delta} - \underline{\delta}_b))
$$

and we infer that

$$
\lim_{N\to\infty} \max_{0\leq n\leq N} |\hat{\epsilon}_n| = 0.
$$

Therefore,

$$
\begin{aligned}
&\lim_{N\to\infty} \max_{0\leq n\leq N} |B(\delta_n^{(b)}) - \hat{B}_n^{(N)}| \\
&= \lim_{N\to\infty} \max_{0\leq n\leq N} |\tilde{\mathbf{h}}(\frac{\pi_b^*}{\nu_1}, R(\delta_n^{(b)})\Phi(\delta_n^{(b)})) - \tilde{\mathbf{h}}(\frac{\pi_b^*}{\nu_1}, \hat{R}_n\hat{\Phi}_n)| \\
&\leq \lim_{N\to\infty} \max_{0\leq n\leq N} K_h |\hat{\epsilon}_n| = 0.
\end{aligned}
$$

■

Implementation details.

Lemma 20. *If $\delta_b^*(\pi_b^*) = \delta_a^*(\pi_a^*) = \delta^*$, then*

$$s(\pi_b^*, \pi_a^*) = \int_{\underline{\delta}}^{\overline{\delta}} \frac{\gamma s + \lambda \tilde{g}(A(x|\pi_a^*))}{\gamma + \lambda(\tilde{g}(1 - B(x|\pi_b^*)) + \tilde{g}(A(x|\pi_a^*)))} f(x)dx$$
$$= 1 + s\left(\frac{r+\gamma}{\delta^* - \overline{\delta}_b(\pi_b^*)}\pi_b^* - F(\overline{\delta}_b(\pi_b^*))\right) - (1-s)\left(\frac{r+\gamma}{\underline{\delta}_a(\pi_a^*) - \delta^*}\pi_a^* + F(\underline{\delta}_a(\pi_a^*))\right)$$

Proof. If $\delta_b^*(\pi_b^*) = \delta_a^*(\pi_a^*) = \delta^*$, $B = B(\cdot|\pi_b^*)$ and $A = A(\cdot|\pi_a^*)$ are distributions separated by δ^*. Therefore,

$$s(\pi_b^*, \pi_a^*) = \int_{\underline{\delta}}^{\overline{\delta}} \frac{\gamma s + \lambda \tilde{g}(A(x))}{\gamma + \lambda(\tilde{g}(1 - B(x)) + \tilde{g}(A(x)))} f(x)dx$$
$$= s\left(\int_{\underline{\delta}}^{\overline{\delta}_b} \frac{\gamma}{\gamma + \lambda\tilde{g}(1 - B(x))} f(x)dx + (F(\underline{\delta}_a) - F(\overline{\delta}_b))\right)$$
$$+ \int_{\underline{\delta}_a}^{\overline{\delta}} \frac{\gamma s + \lambda\tilde{g}(A(x))}{\gamma + \lambda\tilde{g}(A(x))} f(x)dx.$$

Using (23), we can rewrite

$$\int_{\underline{\delta}}^{\overline{\delta}_b} \frac{\gamma}{\gamma + \lambda\tilde{g}(1 - B(x))} f(x)dx = \frac{r+\gamma}{\delta^* - \overline{\delta}_b}\pi_b^*$$

and

$$\int_{\underline{\delta}_a}^{\overline{\delta}} \frac{\gamma s + \lambda\tilde{g}(A(x))}{\gamma + \lambda\tilde{g}(A(x))} f(x)dx = 1 - F(\underline{\delta}_a) - (1-s)\int_{\underline{\delta}_a}^{\overline{\delta}} \frac{\gamma}{\gamma + \lambda\tilde{g}(A(x))} f(x)dx$$
$$= 1 - F(\underline{\delta}_a) - (1-s)\frac{r+\gamma}{\underline{\delta}_a - \delta^*}\pi_a^*.$$

■

Lemma 21. *The equilibrium interdealer price satisfies*

$$P = R(\delta^*) = \frac{\delta^*}{r} + \frac{\gamma}{r}\left(\int_{\delta^*}^{\overline{\delta}} \frac{1 - F(\delta)}{r + \gamma + \lambda\tilde{g}(A(\delta))} d\delta - \int_{\underline{\delta}}^{\delta^*} \frac{F(\delta)}{r + \gamma + \lambda\tilde{g}(1 - B(\delta))} d\delta\right).$$

Proof. The HJB equation (16) evaluated at $\delta = \delta^*$ yields

$$rR(\delta^*) = \delta^* + \gamma \int_{\underline{\delta}}^{\overline{\delta}} (R(\delta) - R(\delta^*))dF(\delta)$$

since $B(\delta) = 1$ for $\delta > \delta^*$ and $A(\delta) = 0$ for $\delta < \delta^*$. Plugging lemma 4 yields

$$rR(\delta^*) = \delta^* + \gamma \int_{\underline{\delta}}^{\overline{\delta}} \int_{\delta^*}^{\delta} \frac{1}{r + \gamma + \lambda(\tilde{g}(1 - B(x)) + \tilde{g}(A(x)))} dx dF(\delta).$$

The proof is then completed by splitting the integral over $\delta < \delta^*$ and $\delta > \delta^*$, applying Fubini theorem, and integrating. ■

D Model implications proofs

D.1 Supply and marginal type proofs

D.2 Dealer competition proofs

D.3 Search frictions and type dynamics proofs

D.4 Liquidity, allocation and welfare proofs

E Voice trading with uniform types

The equilibrium dealer profits are

$$\pi_b^* = \frac{(\delta^* - \overline{\delta}_b)^2}{r + \gamma} \qquad \text{and} \qquad \pi_a^* = \frac{(\underline{\delta}_a - \delta^*)^2}{r + \gamma}.$$

The auxiliary functions β and α, defined in (26) and (27) and solving the Riccati system in proposition 7, admit the closed-form expressions

$$\beta(\delta) = \left(\frac{2\gamma\underline{\delta}_b}{r\xi_\lambda}\right) \frac{1 + e^{-\eta_\lambda + \xi_\lambda\left(\frac{\delta - \underline{\delta}_b}{\underline{\delta}_b}\right)}}{1 - e^{-\eta_\lambda + \xi_\lambda\left(\frac{\delta - \underline{\delta}_b}{\underline{\delta}_b}\right)}}$$

and

$$\alpha(\delta) = \left(\frac{2\gamma(1-\overline{\delta}_a)}{r\xi_\lambda}\right) \frac{1+e^{-\eta_\lambda+\xi_\lambda\left(\frac{\overline{\delta}_a-\delta}{1-\overline{\delta}_a}\right)}}{1-e^{-\eta_\lambda+\xi_\lambda\left(\frac{\overline{\delta}_a-\delta}{1-\overline{\delta}_a}\right)}}.$$

Corollary 2. *If $\psi_1 = 1$ and $F \sim U[0,1]$, the equilibrium allocations are given by*

$$\Phi_1(\delta) = \begin{cases} \frac{\gamma}{\gamma+\lambda} s\delta & \delta < \underline{\delta}_b, \\ s\beta(\delta) & \delta \in \left[\underline{\delta}_b, \overline{\delta}_b\right], \\ s\left(\beta(\overline{\delta}_b) + \delta - \overline{\delta}_b\right) & \delta \in \left(\overline{\delta}_b, \underline{\delta}_a\right), \\ \delta - (1-s)(1-\alpha(\delta)) & \delta \in \left[\underline{\delta}_a, \overline{\delta}_a\right], \\ s - \frac{\gamma s+\lambda}{\gamma+\lambda}(1-\delta) & \delta > \overline{\delta}_a, \end{cases}$$

and

$$\Phi_0(\delta) = \begin{cases} \frac{\gamma(1-s)+\lambda}{\gamma+\lambda}\delta & \delta < \underline{\delta}_b, \\ \delta - s\beta(\delta) & \delta \in \left[\underline{\delta}_b, \overline{\delta}_b\right], \\ (1-s)\left(1-\alpha(\underline{\delta}_a) - (\underline{\delta}_a - \delta)\right) & \delta \in \left(\overline{\delta}_b, \underline{\delta}_a\right), \\ (1-s)(1-\alpha(\delta)) & \delta \in \left[\underline{\delta}_a, \overline{\delta}_a\right], \\ (1-s)\left(1 - \frac{\gamma}{\lambda+\gamma}(1-\delta)\right) & \delta > \overline{\delta}_a. \end{cases}$$

Corollary 3. *If $\psi_1 = 1$ and $F \sim U[0,1]$, the interdealer price is given by*

$$\begin{aligned} P = R(\delta^*) = & \frac{\delta^*}{r} + \frac{\gamma}{r}\left(\frac{1}{2}\left(\frac{1+2\overline{\delta}_a + \overline{\delta}_a^2 - \underline{\delta}_b^2}{r+\gamma+\lambda} + \frac{2(\underline{\delta}_a - \delta^*) - (\underline{\delta}_a^2 - \overline{\delta}_b^2)}{r+\gamma} \right) \right. \\ & \left. + \int_{\underline{\delta}_b}^{\overline{\delta}_b} \frac{\delta}{r+\gamma+\lambda(1-B(\delta))} d\delta + \int_{\underline{\delta}_a}^{\overline{\delta}_a} \frac{1-\delta}{r+\gamma+\lambda A(\delta)} d\delta \right). \end{aligned}$$

The integrals admit explicit expressions in terms of the dilogarithm function.

Corollary 4. *If $\psi_1 = 1$ and $F \sim U[0,1]$, the reservation value is given by*

$$R(\delta) - R(\delta^*) = \begin{cases} -\frac{\underline{\delta}_b - \delta}{r+\gamma+\lambda} - \frac{1}{\gamma}\left(\frac{2\gamma\underline{\delta}_b}{r\xi_\lambda}\right)^2 \beta(\underline{\delta}_b)^{-1} & \delta < \underline{\delta}_b, \\ -\frac{1}{\gamma}\left(\frac{2\gamma\underline{\delta}_b}{r\xi_\lambda}\right)^2 \beta(\delta)^{-1} & \delta \in \left[\underline{\delta}_b, \overline{\delta}_b\right], \\ \frac{\delta-\delta^*}{r+\gamma} & \delta \in \left(\overline{\delta}_b, \underline{\delta}_a\right), \\ \frac{1}{\gamma}\left(\frac{2\gamma(1-\overline{\delta}_a)}{r\xi_\lambda}\right)^2 \alpha(\delta)^{-1} & \delta \in \left[\underline{\delta}_a, \overline{\delta}_a\right], \\ \frac{\delta-\overline{\delta}_a}{r+\gamma+\lambda} + \frac{1}{\gamma}\left(\frac{2\gamma(1-\overline{\delta}_a)}{r\xi_\lambda}\right)^2 \alpha(\overline{\delta}_a)^{-1} & \delta > \overline{\delta}_a. \end{cases}$$