General Equilibrium Under Convex Portfolio Constraints and Heterogeneous Risk Preferences

Tyler Abbot

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Abstract

This paper characterizes the equilibrium in a continuous time financial market populated by heterogeneous agents who differ in their rate of relative risk aversion and face convex portfolio constraints. The model is studied in an application to margin constraints and found to match real world observations about financial variables and leverage cycles. It is shown how margin constraints increase the market price of risk and decrease the interest rate by forcing more risk averse agents to hold more risky assets, producing a higher equity risk premium. In addition, heterogeneity and margin constraints are shown to produce both pro- and counter-cyclical leverage cycles. Beyond two types, more preference types causes a reduction in the severity of crisis and a lower relative deviation from complete markets in almost all variables. Finally, empirical results are given, documenting a novel stylized fact which is predicted by the model, namely that the leverage cycle is both pro- and counter-cyclical.

Keywords: Asset Pricing, Heterogeneous Agents, General Equilibrium, Financial Economics.

Sciences Po, Department of Economics, 28 rue des Saints Pères, Paris, 75007, France
E-mail address: tyler.abbot@sciencespo.fr

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Introduction

Many markets exhibit incompleteness, in the sense that one would like to consume a particular amount in a particular state, but is unable to do so. In financial markets this characteristic is often exhibited through portfolio constraints. Individuals are seldom able to borrow as much as they would like or to short sale stocks without limit. This paper seeks to tackle this problem using a novel solution approach. I characterize the equilibrium of a continuous time financial market populated by agents who differ in their preferences towards risk and face convex portfolio constraints. This equilibrium remains low dimensional by recognizing that one can achieve aggregation results similar to in a complete market. It is then shown through numerical solution how the number of preference types affects financial variables, in particular how diversity can actually reduce the severity of crisis and how preference heterogeneity can produce both pro- and counter-cyclical leverage cycles. Finally, I give qualitative empirical evidence for the applicability of the model to the leverage cycle and document a novel stylized fact about the co-movement between leverage and the aggregate economy.

A fundamental paper by Cvitanić and Karatzas (1992) studied the general case of convex portfolio constraints in partial equilibrium. The authors developed an ingenious way to embed the agent in a series of fictitious economies, parameterized via a sort of Kuhn-Tucker condition, and then to select the appropriate market to make the agent just indifferent. However, their approach was to use convex duality to characterize the solution, which relies on a strict assumption that the relative risk aversion be bounded above by one. This limitation led others to look to solve the primal problem directly, such as He and Pages (1993); Cuoco and He (1994); Cuoco (1997); Karatzas et al. (2003). These works use dense and complex mathematical techniques which may or may not provide tractable solutions for calculation. The present paper takes a more direct approach to solve the primal problem by noticing that homogeneous preferences are associated to a value function which factors into a function of wealth and a function of the aggregate state, under the appropriate ansatz. Using this ansatz, the Hamilton-Jacobi-Bellman equation becomes an ODE over a single state variable. Because of this it is possible to derive a system of equations governing consumption and stochastic discount factors as functions of the same state. This system resembles a Walrasian market and proof of existence of a solution follows identical steps to the classical demand system. Finally, it follows that the minimal state is simply the aggregate dividend, presenting not only a simple mathematical characterization of equilibrium, but also a solution which can feasibly be calculated using standard numerical methods.

This solution approach is studied in an application to margin constraints. Margin constraints are found to increase the market price of risk and decrease the interest rate, contributing to a higher equity risk premium. The interest rate is low because the constraint limits the supply of risk free bonds to the market. This limit in supply pushes up the bond price and down the interest rate. The market price of risk is high because constrained agents are unable to leverage up to take advantage of high returns. On the opposite side of this constraint are risk averse agents who would like to sell their risky assets. They are unable to do so, given the counter-party is the constrained agent. Thus margin constraints create an implicit liquidity constraint which allows the market price of risk to remain high. Finally, asset prices are actually higher than in an unconstrained equilibrium. Constrained agents are unable to purchase more shares of the risky asset, allowing returns to remain high.

Margin constraints and preference heterogeneity generate both pro- and counter-cyclical leverage cycles. When aggregate production is high, less risk averse agents dominate the economy and the price of risky assets is high. High asset prices increase individual wealth and reduce leverage. When aggregate production is low risk averse agents dominate, reducing asset prices. Low asset prices cause individual wealth to be low and individual leverage to be high. With a margin constraint less risk averse agents eventually run into a borrowing limit. This limit forces risk averse agents to hold more risky assets. Returns on these assets must be high in order to compensate the risk averse agents, thus asset prices will be higher under constraint. In turn, total leverage falls. In this way heterogeneous preferences and margin constraints produce both pro- and counter-cyclical leverage cycles. This effect is robust to different assumptions about the distribution of preferences, but the size of the cycle depends on the number of preference types and the level of constraint.

The number of preference types found to reduce the severity of crises for two reasons. First, more preference types implies less severe swings in financial variables even when markets are complete. The marginal agent changes more slowly in the face of aggregate shocks when there is greater heterogeneity. Second, the size of the effect is determined by the mass of agents facing a marginally binding constraint. The effect is large when this mass is large. In particular when a large group of agents arrive at the constraint the supply and demand for credit both shift in, resulting in a credit crunch. This credit crunch is smaller when the mass is spread over several types and when individual constraints bind at different times. In this way preference heterogeneity dampens the effect of negative shocks on margins. This observation has implications for markets in which many diverse investors participate versus markets with a small number of participants, for example the market for index funds versus the market for more complex instruments such as derivative contracts. One can infer that less diverse markets will face larger shifts at points where individual constraints bind. In addition, the model proposed can handle other types of convex portfolio constraints not treated here, as long as these constraints do not directly depend on individual wealth.

Financial leverage has become an important policy variable since the crisis of 2007-2008. In particular leverage allows investors to increase the volatility of balance sheet equity, producing the possibility of greater returns. At the same time leveraged investors are exposed to larger down-side risk. In the face of negative shocks, constrained investors must sell assets to reduce their leverage. This is known as the "leverage cycle". The associated credit contraction produces large volatility in asset prices and has been the target of regulation in the post-crisis era (e.g. the Basel III capital requirement rules). However, whether leverage is pro- or counter-cyclical remains a topic of debate. The equilibrium of the model presented here exhibits both pro- and counter-cyclical leverage cycles, depending on the aggregate state of the economy and the marginal agent. In section 4 I document that cycles are both pro- and counter-cyclical depending on the level of aggregate asset pricing variables which can be interpreted as proxies for marginal preferences. This fact could reconcile some of the empirical debates about the cyclicality of leverage and re-enforces the study of preference heterogeneity as a driver of financial trade.

Convex portfolio constraints arise quite naturally in finance. A convex constraint simply states that the portfolio weights must lie in a convex set containing zero (see Stiglitz and Weiss (1981) for an example of micro-foundations to credit constraints). In macroeconomics there are countless examples of particular models with market incompleteness which can be described in this setting of convex constraints, such as Aiyagari (1994); Kiyotaki and Moore (1997); Krusell and Smith (1998); Bernanke et al. (1999) and many others. This paper's approach could feasibly be applied to those settings, but focuses on a particular application to margin constraints because of its tractability, ubiquity in financial markets, and use in the literature. A margin constraint essentially states that a borrower cannot borrow infinitely against their equity. This type of constraint is seen in consumer finance when borrowing money to purchase a home: one must almost always put up a down payment. In financial markets margin constraints arise in repo markets and other lending vehicles (see Hardouvelis and Peristiani (1992); Hardouvelis and Theodossiou (2002); Adrian and Shin (2010a) for empirical studies of margins). In fact real world experience motivated the theoretical study of leverage cycles initiated by Geanakoplos (1996). In addition limits to arbitrage and financial bubbles have been studied under margin constraints in the context of liquidity (see e.g. Brunnermeier and Pedersen (2009)). Many of these phenomena arise in the model presented in this paper, but the predictions for leverage cycles are emphasized because of their novelty.

In theoretical models leverage cyclicality depends greatly on the underlying assumptions producing trade. In his foundational work on the topic, Geanakoplos (1996) shows how the combination of belief heterogeneity and margin constraints produce a pro-cyclical leverage cycle. However, this finding is in opposition to the contemporary paper by Kiyotaki and Moore (1997), where participation constraints force agents to invest through intermediaries, whose credit constraints produce a counter-cyclical leverage cycle. More recently He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014) also produce counter-cyclical leverage cycles by including intermediaries through whom constrained agents can profit from risky assets. In fact, He and Krishnamurthy (2013) even points out the debate in the applied literature and the fact that, "[Their] model does not capture the other aspects of this process, ... that some parts of the financial sector reduce asset holdings and deleverage." These models imply that the mechanism producing trade determines leverage's cyclicality.

This ambiguity over the cyclicality of leverage has been noted in different ways by the empirical literature. Korajczyk and Levy (2003) study the capital structure of firms and find that leverage is counter-cyclical for unconstrained firms and pro-cyclical for constrained firms. However, Halling et al. (2016) contradict this by showing that target leverage is counter-cyclical onces you account for variation in explanatory variables, pointing out that the effect in Korajczyk and Levy (2003) is only the "direct effect". In the cross section of the economy Adrian and Shin (2010b) find that leverage is counter-cyclical for households, ambiguous for non-financial firms, and pro-cyclical for broker dealers. However, the authors study the relationship between leverage and changes in balance sheet assets. This comparison produces a mechanical correlation which somewhat disappears when assets are replaced by GDP growth as a proxy for the business cycle (see Figure 11). Ang et al. (2011) point out that when accounting for prices broker dealer leverage is counter-cyclical, but that hedge fund leverage is pro-cyclical. These contrary studies can be reconciled when controlling for financial variables such as the price/dividend ratio or the interest rate. In fact, for several sectors studied (see section 4) the leverage cycle is *both* pro- and counter-cyclical. This ambiguity is predicted by the model of preference heterogeneity and margin constraints presented here.

Many authors have criticized the assumption of a representative, constant relative risk aversion agent since Mehra and Prescott (1985) posited the equity risk premium puzzle. The definition of new utility functions was the first major response to this puzzle, in particular Epstein-Zin preferences (Epstein and Zin (1989); Weil (1989)) and habit formation (Campbell and Cochrane (1999)) have been used to explain this and other puzzles. However, several papers have studied preferences across individuals and found them to be heterogeneous and constant in time (Brunnermeier and Nagel (2008); Chiappori and Paiella (2011); Chiappori et al. (2012)), contradicting both of these new branches of the theoretical literature. In addition, Epstein et al. (2014) pointed out that the assumptions necessary to match the risk premium using Epstein-Zin preferences produce unrealistic preference for early resolution of uncertainty. Beyond these criticisms, one needs heterogeneity in order to generate trade at all in any market model. In a representative agent setting one looks for the prices which make the agent indifferent to *not* trading. Risk preference heterogeneity has succeeded in partially responding to these issues.

Heterogeneity in risk preferences has been used to generate trade in financial models since the foundational paper of Dumas (1989). Since then many authors have studied the problem from different angles, assuming different levels of market completeness, utility functions, participation constraints, information structures, etc., but almost always under the assumption of only two preference types (Basak and Cuoco (1998); Coen-Pirani (2004); Guvenen (2006); Kogan et al. (2007); Guvenen (2009); Cozzi (2011); Garleanu and Pedersen (2011); Hugonnier (2012); Rytchkov (2014); Longstaff and Wang (2012); Prieto (2010); Christensen et al. (2012); Bhamra and Uppal (2014); Chabakauri (2013, 2015); Gârleanu and Panageas (2015); Santos and Veronesi (2010)). Cvitanić et al. (2011) studies the problem of N agents with several dimensions of heterogeneity and focuses on the dominant agents, characterizing portfolios via the Malliavan calculus. Abbot (2017) studies a setting with N heterogeneous CRRA agents in a complete financial market using a value function approach and shows how changes in the number of types can produce substantially different quantitative results and how the variance in preferences provides an additional degree of freedom for explaining the equity risk premium puzzle. However, that work produces large amounts of aggregate leverage and high individual margins. This observation points towards the need to introduce some degree of constraint or incompleteness to better match the real world. To that end, this paper studies the same type of economy with N heterogeneous CRRA agents under convex portfolio constraints with an application to margin constraints.

1. A Model of Preference Heterogeneity

1.1. Financial Markets

Consider a continuous time, infinite horizon Lucas (1978) economy with one consumption good. This consumption good, denoted D_t, is produced by a per capita tree[1], whose dividend follows a geometric Brownian motion (GBM):

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dW_t$$

[1] One can either consider N identical trees such that $D_0 = 1.0$ (for example) or one big tree where $D_0 = N$, the two cases are equivalent.

where W_t is a standard Brownian motion and (μ_D, σ_D) are constants. Agents can trade in a (locally) risk-free and a risky security, denoted S_t^0 and S_t respectively. The prices of these securities are assumed to follow an exponential and an Itô process, respectively:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t \qquad (1) \qquad\qquad \frac{dS_t^0}{S_t^0} = r_t dt \qquad (2)$$

where (μ_t, σ_t, r_t) are determined in equilibrium. Individuals are initially endowed with a share in the per-capita tree, α_i.

1.2. Preferences and Wealth

The economy is populated by an arbitrary number N of atomistic agents indexed by $i \in \{1, \ldots, N\}$. Agents have constant relative risk aversion (CRRA) preferences and differ in their rate of relative risk aversion, γ_i, such that their instantaneous utility is given by

$$u_i(c) = \frac{c^{1-\gamma_i}}{1-\gamma_i}$$

Denote by X_{it} an individual's wealth at time t and assume that initial borrowing is zero, which implies an initial wealth $X_{i0} = \alpha_i S_0$. Denote by π_{it} the share of an individual's wealth invested in the risky stock, which implies $1 - \pi_{it}$ is the share invested in the bond. Assuming that trading strategies are self financing, an individual's wealth evolves as

$$dX_{it} = \left[X_{it} \left(r_t + \pi_{it}(\mu_t + \frac{D_t}{S_t} - r_t) \right) - c_{it} \right] dt + X_{it}\pi_{it}\sigma_t dW_t$$

1.3. Portfolio Constraints and Individual Optimization

Individual investors solve a utility maximization problem subject to their self-financing budget constraint and a portfolio constraint:

$$\begin{aligned}
&\max_{\{c_{it},\pi_{it}\}_{t=0}^{\infty}} && \mathbb{E} \int_0^{\infty} e^{-\rho t} \frac{c_{it}^{1-\gamma_i}}{1-\gamma_i} dt \\
&\text{s.t.} && dX_{it} = \left[X_{it} \left(r_t + \pi_{it} \left(\mu_t + \frac{D_t}{S_t} - r_t \right) \right) - c_{it} \right] dt + X_{it}\pi_{it}\sigma_t dW_t \\
& && \pi_{it} \in \Pi_i
\end{aligned}$$

where $\Pi_i \subseteq \mathbb{R}$ represents a closed, convex region of the portfolio space which contains $\{0\}$. For example $\Pi_i = \mathbb{R}$ is the unconstrained case, $\Pi_i = \mathbb{R}^+$ is a short sale constraint, $\Pi_i = \{\pi : \pi \leq m \mid m \geq 0\}$ is a margin constraint. This set is allowed to differ across agents, as implied by the subscript. This paper focuses on an application to margin constraints, but the approach is applicable to any constraint which can be written as a function of the aggregate state[2].

1.4. Equilibrium

Investors are considered to be atomistic and thus I consider a Radner (1972) type equilibrium.

Definition 1. *An equilibrium in this economy is defined by a set of processes* $\{r_t, S_t, \{c_{it}, X_{it}, \pi_{it}\}_{i=1}^N\}$ $\forall$ t, *given preferences and initial endowments, such that* $\{c_{it}, X_{it}, \pi_{it}\}$ *solve the agents' individual optimization problems and the following set of market clearing conditions is satisfied:*

$$\frac{1}{N}\sum_i c_{it} = D_t \ , \ \frac{1}{N}\sum_i (1-\pi_{it})X_{it} = 0 \ , \ \frac{1}{N}\sum_i \pi_{it}X_{it} = S_t \tag{3}$$

I study Markovian equilibria such that the problem can be written as a function of a single state variable[3]. That is for some equilibrium process Y_t, I look for functions $f(\cdot)$ such that $Y_t = f(D_t)$. The existence of such an equilibrium is one of the key mathematical insights of the paper and has implications for the study of other general equilibrium models under constraints.

2. Equilibrium Characterization

To solve this problem I begin with the approach of Cvitanić and Karatzas (1992). This method uses a fictitious, unconstrained economy and a shadow cost of constraint, or Lagrange multiplier, to find the correct pricing process. Unlike in their work I do not use a duality approach, but show how the primal problem can be written in terms of a single state variable. The reason this works is because when utility functions are functionally homogeneous an individual's consumption choice is linear in their wealth and, in turn, their portfolio choice is independent of wealth. In this case the value function factors and the resulting ODE is no

[2]An important limitation of the approach presented here is that it cannot treat constraints which depend explicitly on individual wealth. The reason why will become apparent later, but revolves around the necessity to write equilibrium objects as functions of aggregate variables only.

[3]This characteristic is often suppressed for notational simplicity.

longer a function of individual wealth. Wealth-consumption ratios become the key objects and we can use them in market clearing to derive the solution. This approach will likewise work for any homogeneous utility function, including Epstein-Zin[4]. This process will be described in the following subsections.

2.1. Optimality in Fictitious Unconstrained Economy

In order to find the constrained equilibrium, we define new processes for individual prices, which are "adjusted" by a process ν_{it}, considered the shadow cost of constraint:

$$\frac{dS_{it}^0}{S_{it}^0} = (r_t + \delta_i(\nu_{it}))dt$$
$$\frac{dS_{it}}{S_{it}} = (\mu_t + \nu_{it} + \delta_i(\nu_{it}))dt + \sigma_t dW_t$$

The function $\delta_i(\cdot)$ is the support function of Π_i, which is defined as

$$\delta_i(\nu) = \sup_{\pi \in \Pi_i} (-\nu\pi)$$

In addition, this gives rise to the effective domain of ν_{it} defined by $\mathcal{N}_i = \{\nu \in \mathbb{R} : \delta_i(\nu) < \infty\}$. Finally, we have a complimentary slackness condition which states $\nu_{it}\pi_{it} + \delta_i(\nu_{it}) = 0$. Each agent solves their optimization problem in the face of their individual, fictitious financial market.

Define the stochastic discount factor (SDF) of an individual agent as an Itô process which evolves as a function of the individual's adjustment:

$$\frac{dH_{it}}{H_{it}} = -(r_t + \delta_i(\nu_{it}))dt - \left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right) dW_t \tag{4}$$

By a straight-forward application of the martingale approach (Karatzas et al. (1987)) in this fictitious economy one finds individual consumption as a function of individual SDF's:

$$c_{it} = \left(\Lambda_i e^{\rho t} H_{it}\right)^{-\frac{1}{\gamma_i}} \tag{5}$$

for all i, where Λ_i is the Lagrange multiplier associated to the static budget constraint. In the

[4]In particular, first order conditions from a dynamic program give consumption as $c = u'^{-1}(\partial_X J(X,Y))$, where Y is any arbitrary, aggregate state vector and X an individual's wealth. We would like to find $c = X/V(Y)$. Equate these and rearrange to find $\partial_X J(X,Y) = u'(X/V(Y))$. When preferences are homogeneous of degree $k+1$, $u'(\cdot)$ is homogeneous of degree k. Thus $\partial_X J(X,Y) = u'(1)V(Y)^{-k}X^k$. By integrating with respect to X one finds a proposal for the value function such that consumption is a linear function of wealth.

case where $\Pi_i = \mathbb{R}\ \forall\ i$, the SDF's coincide and the ratios of marginal utilities are constant. However, when agents are constrained in their portfolio choice this is not the case and we have

$$\frac{c_{it}^{-\gamma_i}}{c_{jt}^{-\gamma_j}} = \frac{\Lambda_i H_{it}}{\Lambda_j H_{jt}}$$

These ratios of SDF's, which are proportional to ratios of marginal utilities, are very familiar in the theory of incomplete market equilibria. In Cuoco et al. (2001), a representative agent with state dependent preferences is studied, where the preferences are a weighted average of individual preferences. The stochastic weights are exactly equal to the ratio of marginal utilities. This is also seen in Basak and Cuoco (1998) and Hugonnier (2012). Because ratios of marginal utility are state dependent, the typical approach to aggregation, either through a representative agent or via a market clearing condition is not possible. However, we can make use of the form of preferences to derive aggregation results which govern these state-dependent weights.

2.2. From Partial Equilibrium to Market Clearing

The above results represent partial equilibrium of individual agents. To consider general equilibrium we must aggregate these results across agents, but we have a large number of unknowns, as each individual faces a different SDF. We know that in complete markets under CRRA preferences (or any preferences which are homogeneous of some degree) that the value function factors into a function of wealth and a function of dividends. If this is the case when markets are incomplete, we can cancel wealth from the market clearing conditions. In this spirit, assume an agent's wealth-consumption ratio $V_i(D) = X_{it}/c_{it}$ is a function of the dividend (to be verified in Proposition 5) and substitute into the market clearing conditions in Eq. (3) to derive the following proposition:

Proposition 1. *Assuming the wealth-consumption ratio of an individual agent is given by $V_i(D) = X_{it}/c_{it}$ and that portfolios are deterministic functions of dividends $\pi_i(D) = \pi_{it}$, individual consumption weights $\omega_i(D) = \omega_{it} = c_{it}/D_t$ and stochastic discount factors $H_i(D) = H_{it}$ satisfy a system of coupled non-linear equations given by*

$$1 = \frac{1}{N}\sum_j \left(\frac{\Lambda_j H_j(D)}{\Lambda_i H_i(D)}\right)^{-\frac{1}{\gamma_j}} \omega_i(D)^{\frac{\gamma_i}{\gamma_j}} D^{\frac{\gamma_i}{\gamma_j}-1} \tag{6}$$

$$0 = \frac{1}{N}\sum_j (1-\pi_j(D))V_j(D) \left(\frac{\Lambda_j H_j(D)}{\Lambda_i H_i(D)}\right)^{-\frac{1}{\gamma_j}} \omega_i(D)^{\frac{\gamma_i}{\gamma_j}} D^{\frac{\gamma_i}{\gamma_j}} \tag{7}$$

for all i in $\{1, ..., N\}$.

Proposition 1 implies several things about the solution. Eqs. (6) and (7) are a system of $2N$ equations in $2N$ unknowns. This is a promising fact for solving the model, but how the SDF's enter the equations is problematic. The solution to this system of equations will not be unique since the SDF's enter as ratios. Given a solution, scaling all of the SDF's by the same constant also produces a solution. However, the derivation comes from an application of Walras' law, which reminds us that this represents a price system[5], where H_{it} is an individual's price for consumption. Although it may not be possible to characterize these prices explicitly over the state space, the relative prices of two individuals is pinned down as a function of D. In addition, it is possible to use the standard approach to proving existence of equilibrium in a pricing system to show that the above system has at least one solution.

Proposition 2. *The system of equations Eqs.* (6) *and* (7) *admits at least one solution.*

In addition to this system, we can note that the ratio of SDF's $h_{ijt} = H_{it}/H_{jt}$, is Markov and can be represented as an Itô process. An application of Itô's lemma gives the dynamics as

$$\frac{dh_{ijt}}{h_{ijt}} = \left[\delta_j(\nu_{jt}) - \delta_i(\nu_{it}) + \left(\theta_t + \frac{\nu_{jt}}{\sigma_t}\right)\left(\frac{\nu_{jt} - \nu_{it}}{\sigma_t}\right)\right] dt + \frac{\nu_{jt} - \nu_{it}}{\sigma_t} dW_t \tag{8}$$

There remains only a single dimension of risk and all of the heterogeneity is static, which implies the key variables to find are the adjustments ν_{it}. One can see that when two agents are not constrained, i.e. $\nu_{it} = \nu_{jt} = 0$, the ratio has no drift or diffusion and remains constant, as in complete markets. When one or both agents is constrained the ratio will move depending on how tight is the constraint and the sign of ν_{it} and $\delta_i(\nu_{it})$. Although the individual stochastic discount factor may not be a function only of a low dimensional state, the ratios of SDF's will be and will depend on how the constraints bind and interact with the state variable.

2.3. General Equilibrium Characterization

Equilibrium is characterized by first assuming the existence of a Markovian equilibrium, deriving a system of ODE's for wealth-consumptions ratios, then recovering the adjustments ν_{it} using the complimentary slackness conditions. Given this it is possible to prove optimality of the value functions. First, consider the interest rate and market price of risk:

[5] I must thank Gordon Zitkovic for pointing out this fact.

Proposition 3. *The interest rate and market price of risk can be shown to be functions of weighted averages of individuals' consumption weights, preference parameters, and adjustments such that*

$$\theta_t = \frac{N}{\sum_i \frac{\omega_{it}}{\gamma_i}} \left(\sigma_D - \frac{1}{\sigma_t N} \sum_i \frac{\omega_{it}\nu_{it}}{\gamma_i} \right) \tag{9}$$

$$r_t = \frac{N}{\sum_i \frac{\omega_{it}}{\gamma_i}} \left(\mu_D + \frac{\rho}{N} \sum_i \frac{\omega_{it}}{\gamma_i} - \frac{1}{N} \sum_i \frac{\omega_{it}}{\gamma_i} \delta_i(\nu_{it}) \right. \tag{10}$$

$$\left. - \frac{1}{2N} \sum_i \frac{1+\gamma_i}{\gamma_i^2} \left(\theta_t + \frac{\nu_{it}}{\sigma_t} \right)^2 \omega_{it} \right) \tag{11}$$

The interest rate and market price of risk take a typical form, but are augmented by the adjustment to individuals' marginal utilities. First notice that the market price of risk (Eq. (9)) is determined by the fundamental volatility σ_D divided by the weighted average of elasticity of intertemporal substitution (EIS), exactly as in complete markets (Abbot (2017)). In addition the constraint will either increase or reduce the market price of risk, depending on the domain of ν_{it}. In the case of margin constraints $\nu_{it} \leq 0$, so the market price of risk will be weakly higher under constraint. This is driven by an implicitt liquidity constraint. When less risk averse agents are constrained they are unable to take advantage of high returns. In addition, the effect of volatility implies that in times when stock price volatility is low, greater constraint implies greater returns. This correlation is again driven by the fact that agents cannot borrow to take advantage of the returns, producing the same type of liquidity effect described in the limits-to-arbitrage literature (e.g. Brunnermeier and Pedersen (2009) or Hugonnier (2012)). Risk neutral agents would arbitrage away the high returns, but cannot because of their margin constraint.

The interest rate similarly exhibits a familiar shape. We see a rate of time preference term, an intertemporal smoothing term, and a prudence or risk preference term:

$$r_t = \underbrace{\rho}_{\text{Rate of Time Preference}} + \underbrace{\frac{\mu_D - \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i}\delta_i(\nu_{it})}{\frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i}}}_{\text{Intertemporal Smoothing}} - \underbrace{\frac{1}{2} \frac{\frac{1}{N}\sum_i \frac{1+\gamma_i}{\gamma_i^2}\left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right)^2 \omega_{it}}{\frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i}}}_{\text{Prudence/Risk Preferences}}$$

Both the intertemporal smoothing and prudence terms are augmented by the constraint. Under a homogeneous margin constraint, $\delta_i(\nu_{it}) = -m\nu_{it}$, but recall that $\nu_{it} \leq 0$, which together imply that the constraint reduces interest rates through the intertemporal smoothing term. Constrained agents are unable to supply bonds to the market in order to transfer consumption and wealth from the future to today. A lower supply of bonds pushes up the price and down the interest rate. At the same time constraint affects the interest rate through the prudence motive by changing the demand for precautionary savings. Individuals demand more precautionary savings when their SDF is more volatile (Kimball (1990)). When agents are constrained, their SDF is less volatile as they are unable to increase their exposure to fundamental risk. Ceterus paribus, this reduces the demand for precautionary savings and increases the interest rate, counteracting the intertemporal motive. Together these forces produce an equity risk premium which depends on the shape of heterogeneity, the degree of constraint, and the state variable, all driven by the individual consumption weights which determine the marginal agents.

How consumption weights evolve over time is important not only from an economic perspective, but also in order to derive the solution of the model. We can study the dynamics of consumption weights by applying Itô's lemma and matching coefficients to find their drift and diffusion:

Proposition 4. *Consumption weights follow an Itô process whose dynamics are given by:*

$$\frac{d\omega_{it}}{\omega_{it}} = \mu_{\omega it}dt + \sigma_{\omega it}dW_t$$

where

$$\mu_{\omega it} = \frac{1}{\gamma_i}\left(r_t + \delta_i(\nu_{it}) - \rho + \frac{1}{2}\frac{1+\gamma_i}{\gamma_i}\left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right)^2 - \sigma_D\left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right)\right) + \sigma_D^2 - \mu_D \tag{12}$$

$$\sigma_{\omega it} = \frac{1}{\gamma_i}\left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right) - \sigma_D \tag{13}$$

These equations are very similar to those one finds in the complete markets case (Abbot (2017)), but augmented by the constraint. In particular, consider the volatility of consumption weights given in Eq. (13). An agent's consumption volatility is exactly zero when their preference parameter satisfies

$$\gamma_i = \frac{1}{\xi_t} - \frac{\Xi_t}{\xi_t}\frac{1}{\sigma_D} + \frac{\nu_{it}}{\sigma_D\sigma_t} \quad \text{where} \quad \xi_t = \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i} \quad , \quad \Xi_t = \frac{1}{N}\sum_i \frac{\omega_{it}\nu_{it}}{\gamma_i}$$

We can think of this as the marginal preference level in the market for consumption. However, it is possible that this preference level is not unique. Consider the case where some agents face a margin constraint, but others do not. Amongst the unconstrained agents, the marginal preference level corresponds to the first two terms, while among the constrained agents all of the terms matter. Given $\nu_{it} \leq 0$ under margin constraints, there could very well exist both a constrained and an unconstrained agent who has zero consumption volatility. This is driven by the constrained agents being unable to leverage up to gain more exposure to aggregate risk.

Since θ_t and r_t are functions of $\{\omega_{it}\}_{i=1}^N$ and since these are functions of D_t, it remains to show that $\{\nu_{it}\}_{i=1}^N$ is as well in order to show that an equilibrium with only D_t as a state variable is appropriate. First, one can derive a system of ODE's for individual wealth/consumption ratios, from which one can determine the adjustments.

Proposition 5. *Given Propostions 1 and 3 and assuming adjustments and volatility are functions of dividends such that $\nu_i(D) = \nu_{it}$ and $\sigma(D) = \sigma_t$, it is possible to define the interest rate and market price of risk as functions of dividends such that $r_t = r(D)$ and $\theta_t = \theta(D)$. Assuming there exists a Markovian equilibrium in D_t, the individuals' wealth-consumption ratios, $V_i(D) = X_{it}/c_{it}$, satisfy ODE's given by*

$$\begin{aligned}\frac{\sigma_D^2 D^2}{2} V_i''(D) + \left[\frac{1-\gamma_i}{\gamma_i}\left(\theta(D) + \frac{\nu_i(D)}{\sigma(D)}\right)\sigma_D + \mu_D\right] D V_i'(D) \\ + \left[(1-\gamma_i)(r(D) + \delta_i(\nu(D))) - \rho + \frac{1-\gamma_i}{2\gamma_i}\left(\theta(D) + \frac{\nu_i(D)}{\sigma(D)}\right)^2\right]\frac{V_i(D)}{\gamma_i} + 1 = 0\end{aligned} \tag{14}$$

which satisfy boundary conditions

$$\lim_{D \to D^*} V_i(D) = \frac{\gamma_i}{\rho - (1-\gamma_i)\left(\frac{(\theta(D^*) + \nu_i(D^*)/\sigma_D)^2}{2\gamma_i} + r(D^*) + \delta(\nu_i(D^*))\right)} \quad \textit{for } D^* \in \{0, \infty\} \tag{15}$$

These ordinary differential equations represent the shape of individuals' wealth/consumption ratios over the state space. Unlike in complete markets, however, the system is highly non-linear, since the coefficients depend in a complicated way on the solution itself. This will make the solution to the problem quite challenging from a numerical perspective, but the low dimension will at least provide a degree of tractability.

Next, consider the portfolios of individuals, given in Proposition 6.

Proposition 6. *Assuming adjustments and volatility can be written as functions of dividends such that $\nu_i(D) = \nu_{it}$ and $\sigma(D) = \sigma_t$, it can be shown that portfolios are functions of dividends such that $\pi_i(D) = \pi_{it}$, where*

$$\pi_i(D) = \frac{1}{\gamma_i \sigma(D)}\left(\theta(D) + \gamma_i \sigma_D D \frac{V_i'(D)}{V_i(D)} + \frac{\nu_i(D)}{\sigma(D)}\right) \tag{16}$$

One can see right away that portfolios take the typical ICAPM form (Merton (1971)). There is first a myopic term, represented by the market price of risk scaled down by risk aversion and volatility, which gives the instantaneous portfolio demand of an individual given the market price of risk. Next is a hedging term, determined by the co-movement of an individual's wealth with the aggregate state. Finally, there is a constraint term, which compensates the individual's portfolio such that they are within the constraint set.

On an aggregate level, we can derive asset pricing variables from an application of Itô's lemma and from market clearing for wealth.

Proposition 7. *Assuing adjustments can be written as functions of dividends such that $\nu_i(D) = \nu_{it}$, it can be shown that volatility and the price dividend ratio are functions of dividends such that $\sigma(D_t) = \sigma_t$ and $\mathcal{S}(D) = S_t/D_t$, where*

$$\sigma(D) = \sigma_D + \frac{1}{N}\sum_i \omega_i(D)\left[V_i(D)\sigma_{\omega i}(D) + V_i'(D)\sigma_D D\right]/\mathcal{S}(D) \tag{17}$$

and

$$\mathcal{S}(D) = \frac{1}{N}\sum_i \omega_i(D)V_i(D) \tag{18}$$

Volatility in Eq. (17) is driven by the fundamental volatility, the shape of wealth consumption ratios, and the volatility of consumption weights. When agents have high volatility in consumption weights, the volatility of asset prices will be higher. At the same time, individuals' wealth will be less volatile under constraint. This will produce a reduction in volatility. We will see these two forces in the numerical simulations in section 3.

Finally we need to derive an expression for $\{\nu_{it}\}_{i=1}^N$ in order to close the model. The functional form depends on the type of constraint. To that end, I will focus from here only on margin constraints. The following proposition gives the functional form for the adjustments under homogeneous margin constraints when $\pi_{it} \leq m$ for all i, where $m \geq 0$, which implies an effective domain of $\mathcal{N}_i = \{\nu : \nu \leq 0\}$ and a support function of $\delta_i(\nu) = -m\nu$.

Proposition 8. *Under margin constraints, adjustments can be written as functions of dividends such that $\nu_i(D_t) = \nu_{it}$, where*

$$\nu_i(D) = \min\left\{0; m\gamma_i\sigma(D)^2\left(1 - \frac{1}{m\sigma(D)}\left(\frac{\theta(D)}{\gamma_i} + \sigma_D D\frac{V_i'(D)}{V_i(D)}\right)\right)\right\} \tag{19}$$

Here one can see clearly where the kink which produces the endogenous volatility jump previously mentioned. The point where an agents constraint binds will produce a kink in their adjustment, a kink in their constraint, and in turn a kink in their consumption choice.

The equations in Proposition 8 essentially close the model. Given the above propositions, it is possible to prove that this is indeed an equilibrium.

Proposition 9. *Suppose there exist bounded positive functions $V_i(D) \in C^1[0,\infty) \cup C^2[0,\infty)$ that satisfy the system of ODE's in Eq.* (14) *and boundary conditions Eq.* (15)*. Additionally, assume the processes θ_t, ν_{it}/σ_t, σ_t, and π_{it} are bounded and that $|\sigma_t| > 0$. Then there exists a Markovian equilibrium satisfied by Propostions 4, 5 and 8*

This is not to say that this is the only equilibrium, as one could characterize a higher dimensional equilibrium taking consumption weights as state variables, as is done in Chabakauri (2015) or Gârleanu and Panageas (2015) for two agents. However, what this equilibrium shows is that it is possible to solve the model in a lower dimensional space. Because all of the risk in the economy is driven by a single Brownian motion and because of the simple functional form provided by CRRA preferences, we do not need to resort to higher dimensional methods and this equilibrium should hold even in the continuum, as $N \to \infty$.

3. Numerical Solution

This section presents numerical results for several assumptions about the distribution of preferences. First, the case of two types is evaluated and the cyclicality of the leverage cycle is emphasized. The leverage cycle is pro- or counter-cyclical depending on the marginal agent. In addition, the severity of cycles depends on whether the least risk averse agent is constrained. This solution is discussed in relation to previous solutions with two types, emphasizing the difference created by the choice of state variable. Second, I study increasing the number of types over a given support and find that heterogeneity reduces the severity of deviations from complete markets. This observation implies that the number of types is important when considering the effect of margin constraints as a policy tool. Over all simulations I hold fixed $(\mu_D, \sigma_D, \rho) = (0.01, 0.032, 0.02)$, chosen to compare to Chabakauri (2015).

3.1. Two Types and Leverage Cycles

Consider the case of two agents with relative risk aversion $(\gamma_1, \gamma_2) = (1.5, 3.0)$ facing a margin constraint of $m = 1.3$. This corresponds to a constraint on individuals' financial leverage of $Debt/Equity = |1 - \pi_{it}|/\pi_{it} \leq 0.2$. This constraint produces substantial deviations from complete markets. The less risk averse agent's constraint binds over the lower part of the state space. In this region the equity risk premium is high and volatility low. However, volatility is higher in the unconstrained region and this excess volatility rises as the economy approaches the constraint from the right. In addition, at the point where the less risk averse agent's constraint binds a singularity forms, in particular consumption weights are kinked.

Finally, the peak of leverage is reduced and the inflection point shifts out, pushing the change in cyclicality to a higher level of per-capita consumption.

Consider first portfolios in this economy. Figure 1 plots portfolios for the two individuals over a truncated portion of the state space[6] As you can see the agents both leverage up in both complete and incomplete markets as dividends fall. This is driven by an improvement in the investment opportunity set (see Figure 3 to be discussed below). As dividends fall the most risk averse agent dominates and returns must rise to make them indifferent, improving investment opportunities. However, the least risk averse agent encounters the constraint and the most risk averse agent seems to leverage up. What happens around the point where the constraint binds is particularly interesting.

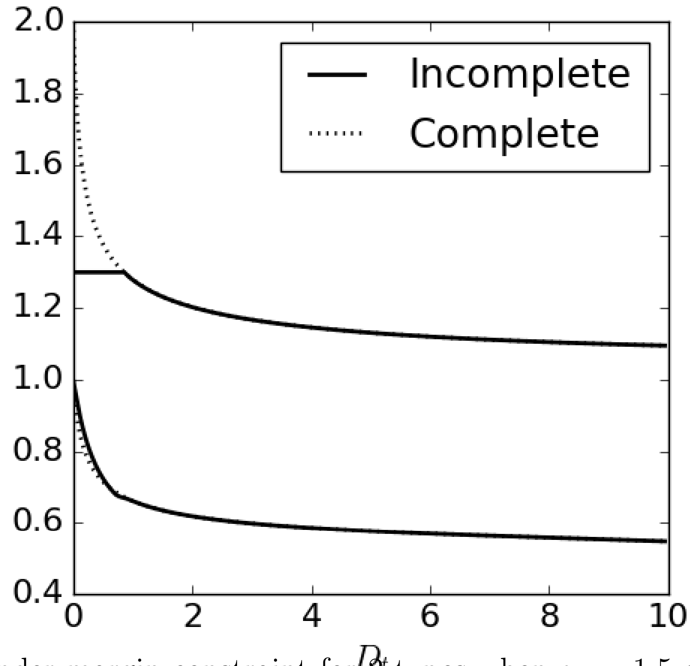


Fig. 1. Portfolios under margin constraint for 2 types when $\gamma_1 = 1.5$ and $\gamma_2 = 3.0$, zoomed into the singular point.

Figure 2 plots portfolios for the least (left plot) and most (right plot) risk averse agents separately. The least risk averse agent holds essentially the same portfolio in the unconstrained region and does not change their strategy. On impact their portfolio exhibits a kink as they are forced to hold the constrained portfolio. The more risk averse agent on the other hand anticipates the constraint. Before impact they begin to reduce (relatively speaking) their leverage and after impact they continue to move further from the complete markets optimum. Finally, the slope of their portfolio picks up and they begin to leverage

[6]Plots are presented truncated as the support is semi-infinite and most of the curvature is in the lowest section.

up, eventually over taking the complete markets solution. These dynamics are driven by changes in the investment opportunity set, which drive the individuals' choices.

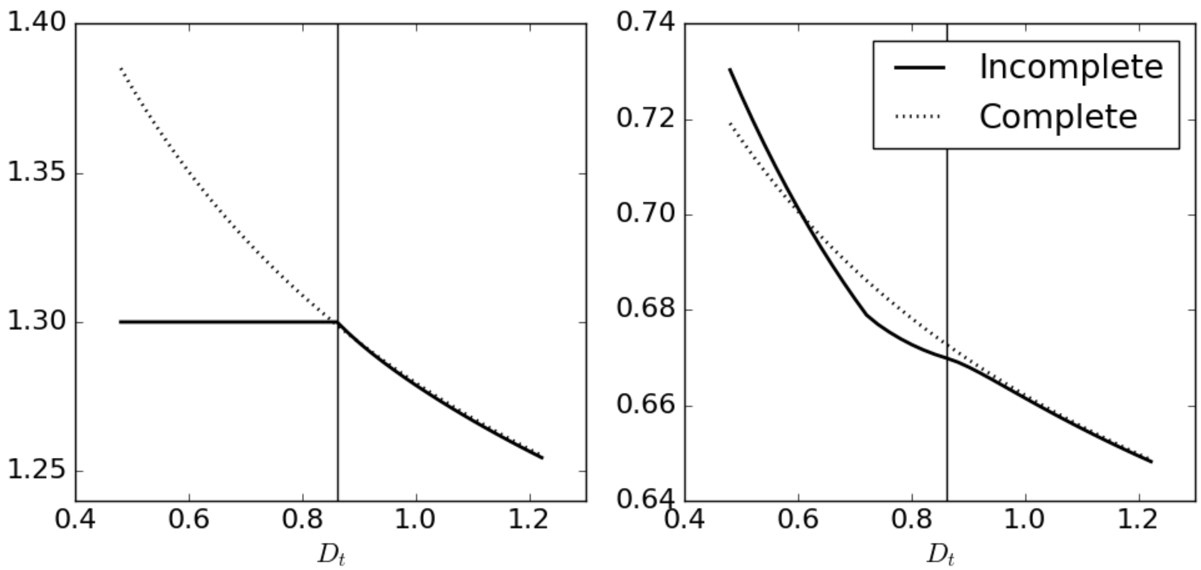


Fig. 2. Portfolios under margin constraint for 2 types when $\gamma_1 = 1.5$ (left plot) and $\gamma_2 = 3.0$ (right plot), zoomed into the singular point, which is represented by the vertical line.

The key financial variables representing the investment opportunity set are the interest rate, market price of risk, dividend yield, and volatility. Figure 3 plots these financial variables in levels for complete and incomplete markets. In both cases the risk averse agent dominates as dividends fall, pushing up asset prices, the interest rate, and the market price of risk. Lower values of the dividend also correspond to times of greater trade between the two types, increasing volatility. In the constrained region the investment opportunities are better under constraint, as interest rates are lower, the market price of risk higher, and volatility substantially lower. In addition we can observe a clear smile in volatility. However it is difficult to tell by how much the constraint is affecting the outcome.

These effects are more clearly seen in deviations from complete markets (i.e. $(Y_I/Y_c - 1) \times 100\%$), shown in Figure 4. Volatility is substantially lower under constraint, but in the unconstrained region is in fact slightly higher. This is driven by a steeper wealth/consumption ratio under constraint, even in the unconstrained region. The market price of risk is over 5% higher in the constrained region to compensate risk averse agents for holding a larger share of the risky asset. One would think this would push down asset prices, as risk averse agents have a lower autarky price for risky assets, but in fact the dividend yield is lower (and the price/dividend ratio thus higher) under constraint. Less risk averse agents are limited by their constraint from participating and arbitraging away the high returns. This limit to

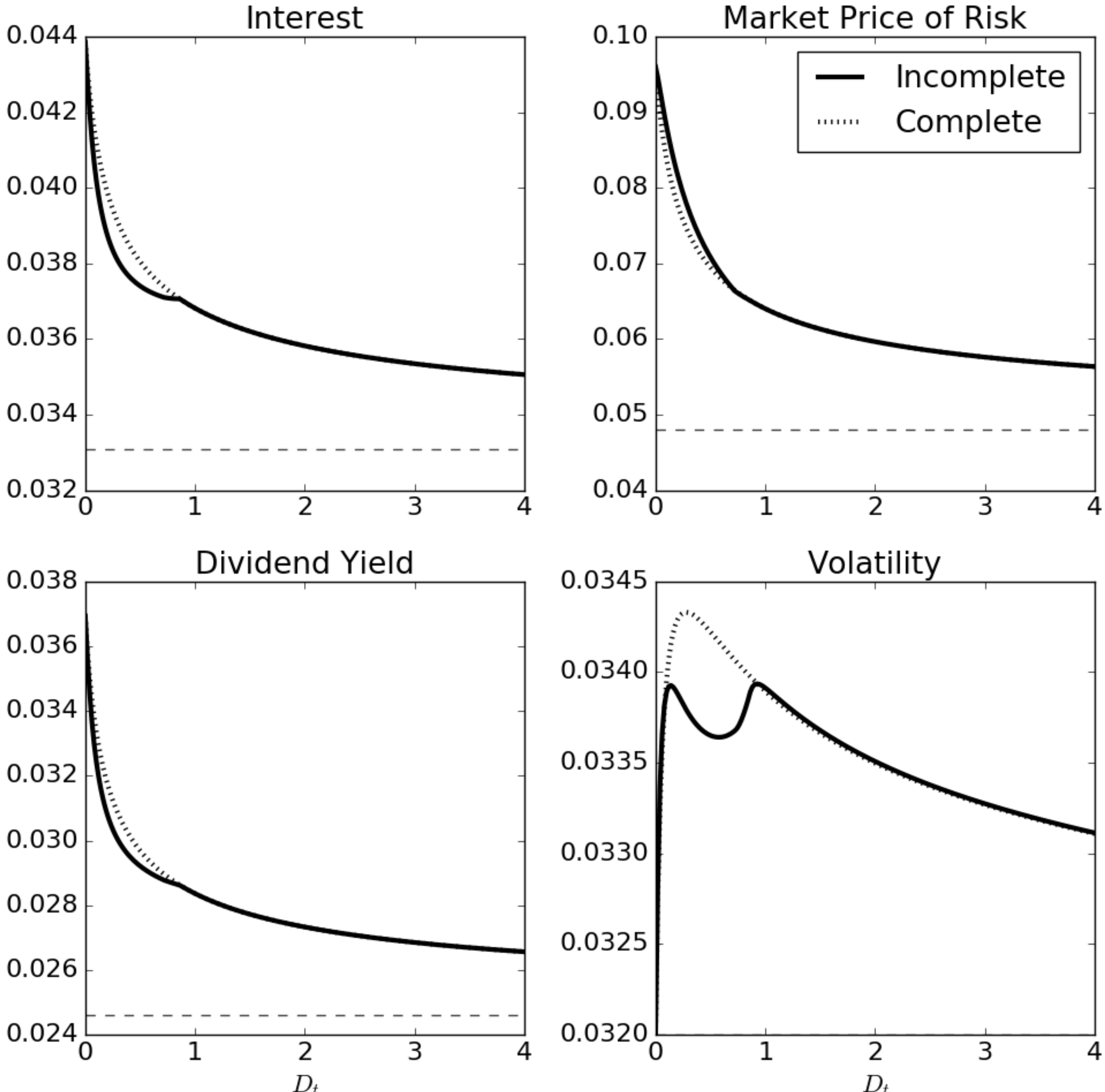


Fig. 3. Financial variables under margin constraint for 2 types when $\gamma_1 = 1.1$ and $\gamma_2 = 3.0$.

arbitrage is similar to the type of liquidity constraints which are posited in other settings (e.g. Brunnermeier and Pedersen (2009)). Asset prices are above their fundamental value because of this limit to risk neutral agents' ability to profit form the arbitrage and a financial bubble arises (as in Hugonnier (2012)). Despite this bubble, risk averse agents see the reduced volatility and the increase in the market price of risk and shift wealth to the risky asset. This produces a contraction in the supply of credit which pushes down the interest rate and affects leverage.

The effect on leverage is substantial given both a supply effect and a demand effect. Figure 5 plots several measures of leverage and the leverage cycle. In the constrained region

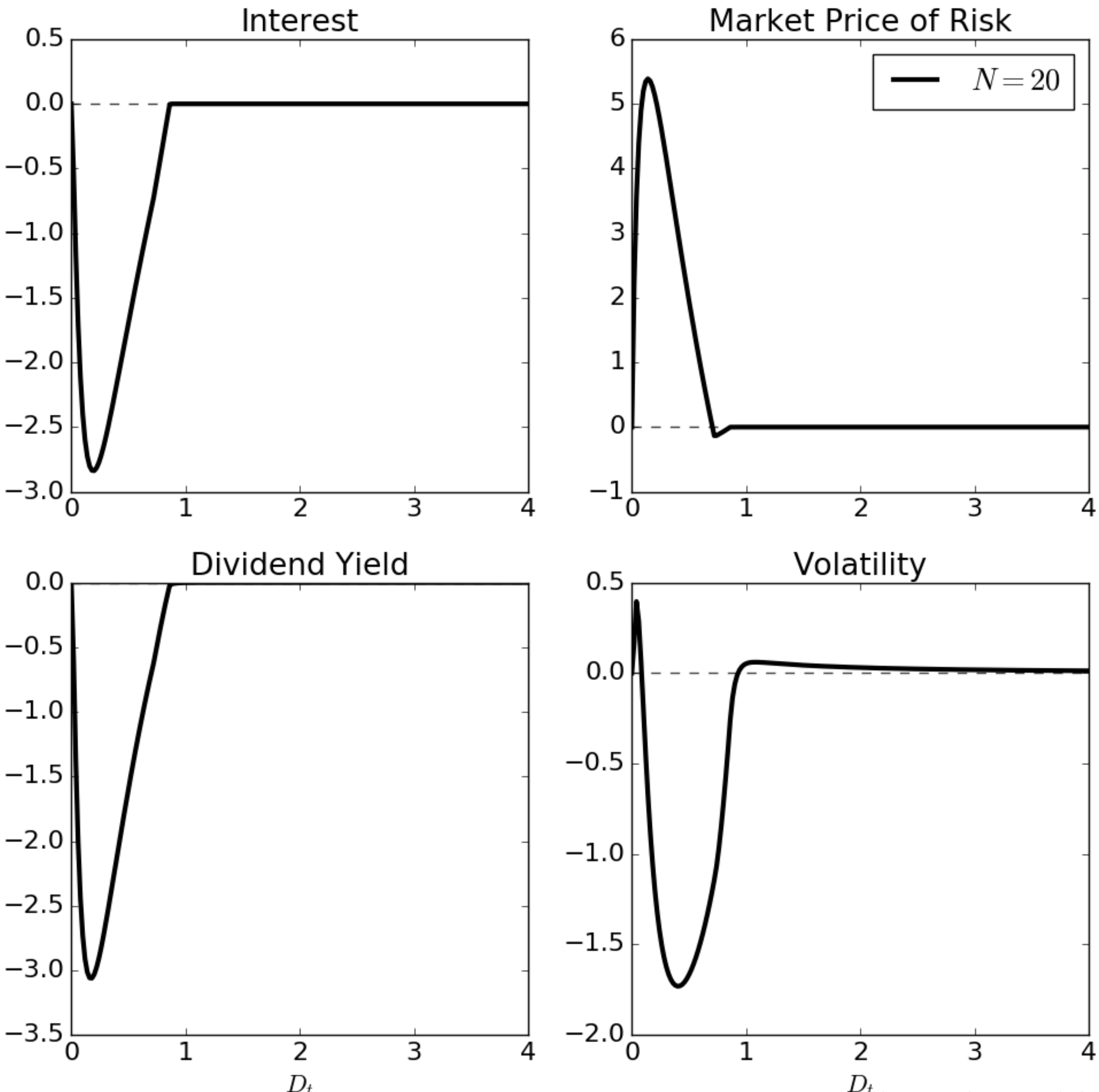


Fig. 4. Financial variables in percent deviations from complete markets $(100 \times (Y_I - Y_c)/Y_c)$ under margin constraint for 2 types when $\gamma_1 = 1.1$ and $\gamma_2 = 3.0$.

leverage is lower than if markets were complete, driven by both demand and supply forces. The demand for credit is artificially lower under constraint when risk neutral agents cannot leverage up. The supply of credit is also reduced because risk averse agents shift wealth into risky assets. They do so because they see low volatility and high expected returns. At the point where the constraint binds, leverage is kinked as supply jumps. At the inflection point, demand for credit is held down by the constraint, but risk averse agents actually leverage up. They see a jump reduction in volatility which produces an increase in expected returns on risky assets. Risk averse agents shift wealth into risky shares and the supply of credit contracts. However they may borrow to purchase more shares, possibly replacing the missing demand for credit. This produces an ambiguous effect on the demand for credit and on the interest rate at the boundary when constraint binds. However, there is an unambiguous change in the cyclicality of leverage.

Leverage cycles are both pro- and counter-cyclical in both complete and incomplete markets, but the dynamics of this cyclicality is vastly different under the two regimes. In complete markets, the slope of leverage varies smoothly, moving from positive to negative as one moves through the state space. Only in very bad states does leverage exhibit pro-cyclicality, as risk averse agents begin to dominate and the interest rate becomes too high for risk neutral agents to desire to borrow. This inflection point becomes a singularity under margin constraints. In Figure 5(b) we can see a jump at the interface between the constrained and the unconstrained region. At the interface a small negative shock causes the economy to jump from counter- to pro-cyclical leverage cycles because of the formation of a bubble and the artificial limit on borrowing. In Figure 5(c) we see that the constraint causes the inflection point to shift out substantially over values of the price/dividend ratio. Although only a small portion of the state space is constrained, the constraint is in effect for a very large portion of the range of asset prices achieved by the economy. These observations will be studied empirically in section 4.

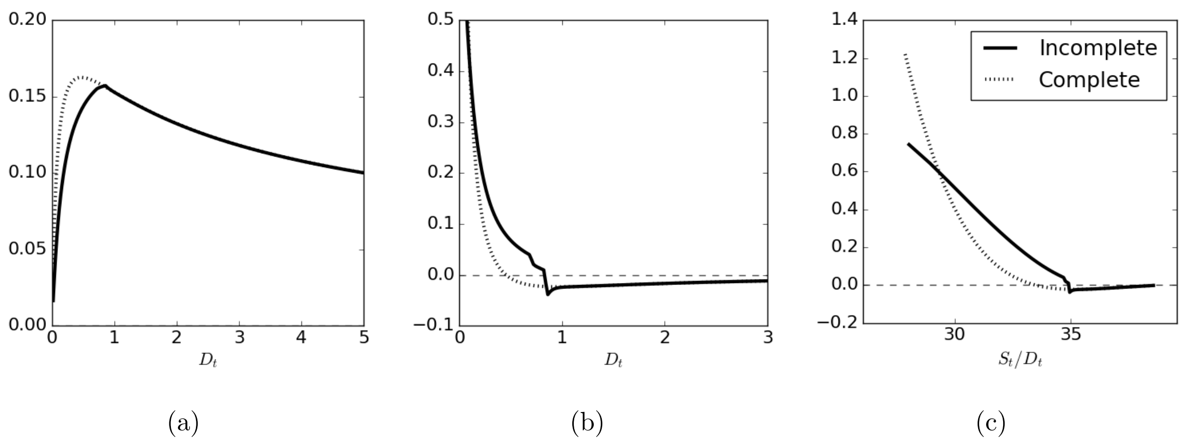
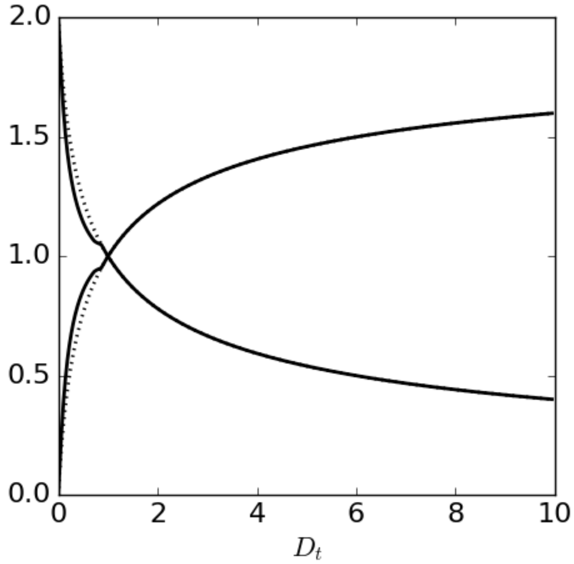
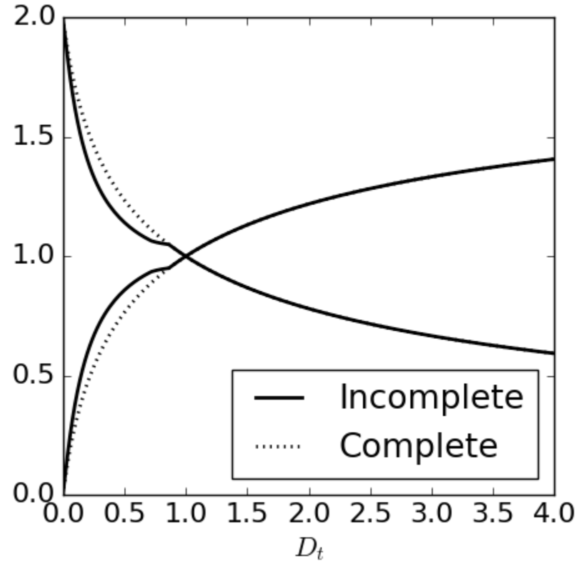


Fig. 5. Leverage for two agents when $(\gamma_1, \gamma_2) = (1.1, 3.0)$. Figure 5(a) plots aggregate financial leverage, Figures 5(b) and 5(c) plots $\partial_D \; Lev$ as a function of D_t and S_t/D_t, respectively.

Typically one considers two agents and takes the consumption shares of one individual as the state variable when studying with heterogeneous preferences (e.g. Chabakauri (2013, 2015); Gârleanu and Panageas (2015) etc.). This is convenient and intuitive from a modeling perspective and gives results which imply binding constraints over a substantial portion of the state space. However, this share is not readily observable in economic data. The equilibrium presented above is in terms of an observable quantity and the numerical solution shows that the constrained portion of the state space is actually rather small. In Figure 6 we can see that the effect of constraint on consumption weights is small for two types. Contrast this to if we were considering consumption weights themselves as a state variable, in which case it would seem that half of the state space is constrained. We see that the constraint binds only in very bad times when characterizing the equilibrium over D_t, .



(a)



(b)

Fig. 6. Consumption weights (ω_1, ω_2) for two types when $(\gamma_1, \gamma_2) = (1.1, 3.0)$, over a broader (Figure 6(a)) and tighter (Figure 6(b)) portion of the state space.

3.2. Increasing Heterogeneity and The Severity of Crisis

In order to consider what happens when the degree of heterogeneity increases, consider simulating 2, 5, 10, and 20 agents, but now assume their preferences are evenly spaced over $[1.5, 3.0]$. Increasing the number of preference types has a surprising effect on the severity of crises and the amount to which the constraint affects market outcomes. As the number of types increase, the deviation of the economy from the complete markets economy goes to zero for all aggregate variables except leverage and the market price of risk.

In Figure 7 you can see relative deviations of financial variables from the complete markets case. First, with intermediate preference types there is more liquidity in the market, increasing the market-clearing level of portfolio weights. This increase causes the constraint to bind for every agent at a higher level of the dividend, which in turn causes the point where the first constraint binds to be further to the right. However, the deviation is smaller for all variables. This is because each individual agent has a smaller weight in the economy and their constraint matters less. At the same time, the market price of risk remains high. As before, the returns on risky assets must be higher to compensate risk averse agents for holding a larger share, but by spreading agents out over the support of preferences, the severity of crisis is reduced. In this way, preference heterogeneity and constraint generate a limit to arbitrage. There exists excess returns from which less risk averse agents are unable to profit because of margin constraints and these excess returns are incorporated mainly into the market price of risk as the number of agents grows.

In addition, the cyclicality of leverage is affected by the number of types. Figure 8 presents the aggregate leverage in each simulation and Figure 9 presents the deviations from complete markets. The constrained region exhibits pro-cyclical cycles over more states when there are more types. The level of leverage becomes more stable with more types, the peak of leverage is reduced, and the slope of leverage with respect to the state is smaller in absolute value. In this way, greater heterogeneity reduces the severity of the leverage cycle. As dividends fall, agents leverage up until they hit their constraint, at which point they begin to deleverage If a large mass hits this constraint at the same time, the resulting cycle will be greater, while if smaller groups are constrained at heterogeneous points in the state space the peak is reduced.

The degree of heterogeneity has a substantial effect on model outcomes when agents face margin constraints. The combination of heterogeneity and margin constraints reconciles several facts about the financial market. With only two types there is a drop in volatility at the constraint threshold, while with many types this is reduced. In addition, this reduction in volatility is associated with a change in the cyclicality of the leverage cycle. The market price of risk is greater when agents are constrained and this fact remains even when the number of types grows. There is a limit to arbitrage as risk neutral agents cannot borrow to profit from higher returns, producing a bubble whose size falls when there are many types, despite returns remaining high. We can investigate the cyclicality of leverage as an empirical implication of the model presented above and compare across different models which one might consider more or less diverse.

4. The Cyclicality of Leverage

Beliefs driven leverage cycles are pro-cyclical according to theory. This implication is somewhat contradicted in several empirical studies, including Adrian and Shin (2010b). In

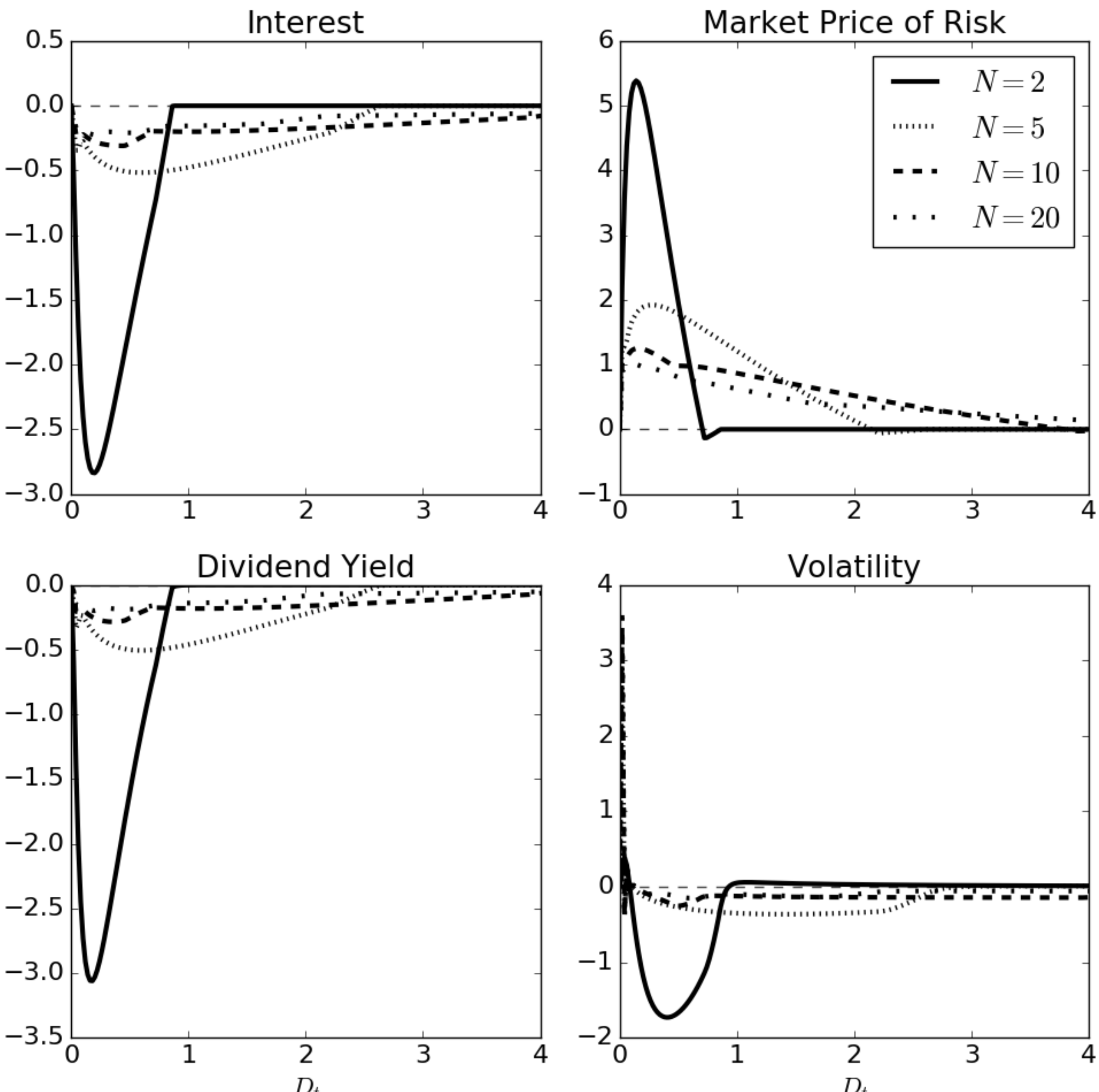


Fig. 7. Relative deviation of financial variables under margin constraint from complete market for 2, 5, 10, and 20 agents evenly distributed over $[1.1, 3.0]$.

that paper the authors note that the leverage cycle is only pro-cyclical for a particular sector of the economy, asset broker/dealers. However, those authors plot leverage as a function of total assets, which produces a mechanical correlation. Consider the definition of financial leverage:

$$Leverage = \frac{Liabilities}{NetWealth} = \frac{Liabilities}{Assets - Liabilities}$$

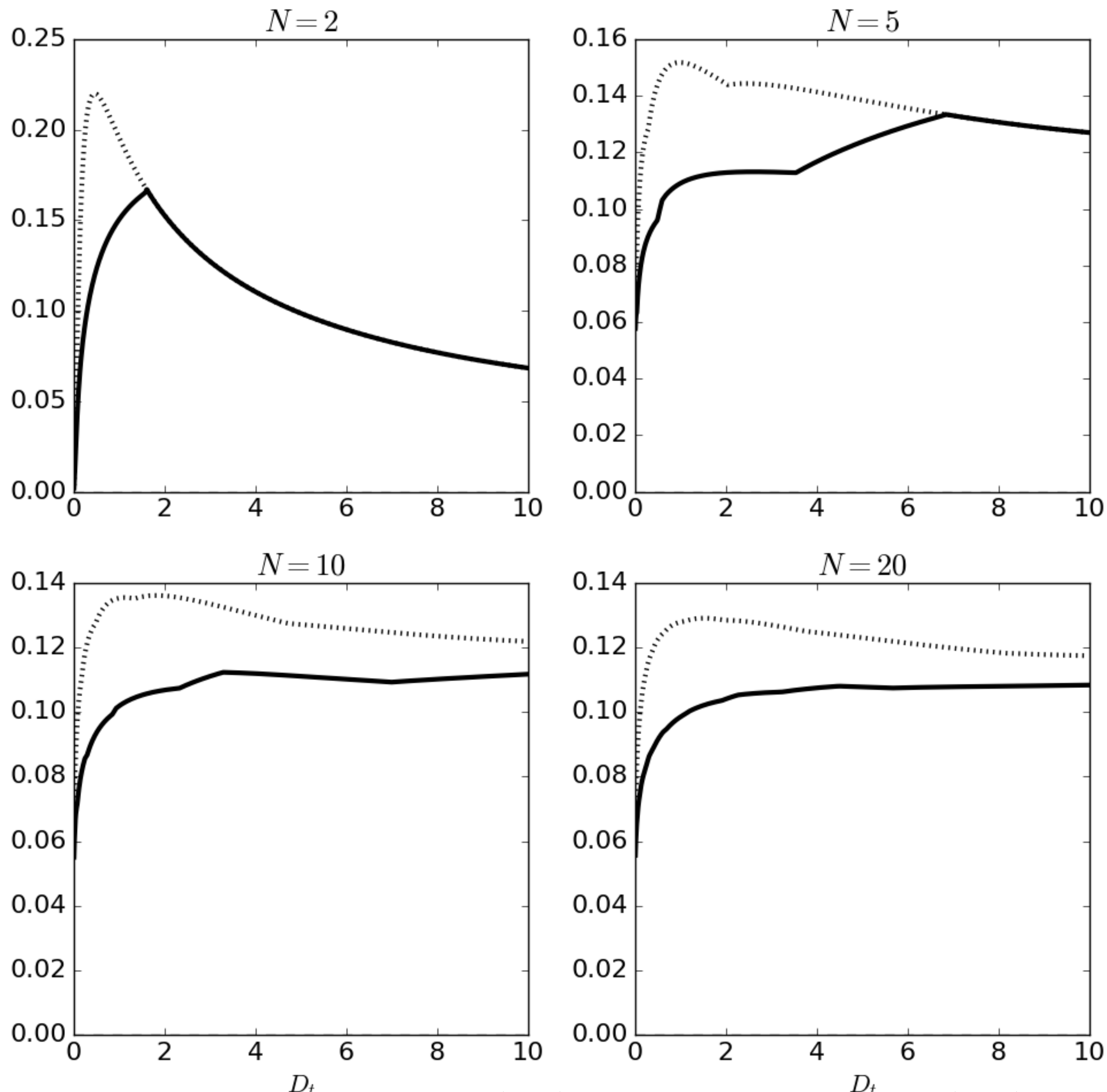


Fig. 8. Leverage under complete market (dots) and under margin constraint (solid) for 2, 5, 10, and 20 agents evenly distributed over $[1.1, 3.0]$.

Increases in balance sheet assets produce a negative correlation between leverage and assets[7](Ang et al. (2011)). Figure 10(a) plots the rate of growth in leverage against the rate of growth in assets for all sectors over 1952Q1 to 2017Q1 as measured from the US Flow of Funds. As you can see, there is a clear negative relationship.

Consider instead changes in GDP as a proxy for the business cycle. Figure 10(b) plots the rate of growth in leverage for all sectors against the rate of growth in GDP over the same

[7]However, this makes the fact that Adrian and Shin (2010b) find pro-cyclical leverage cycles for broker/dealers all the more substantial of a finding

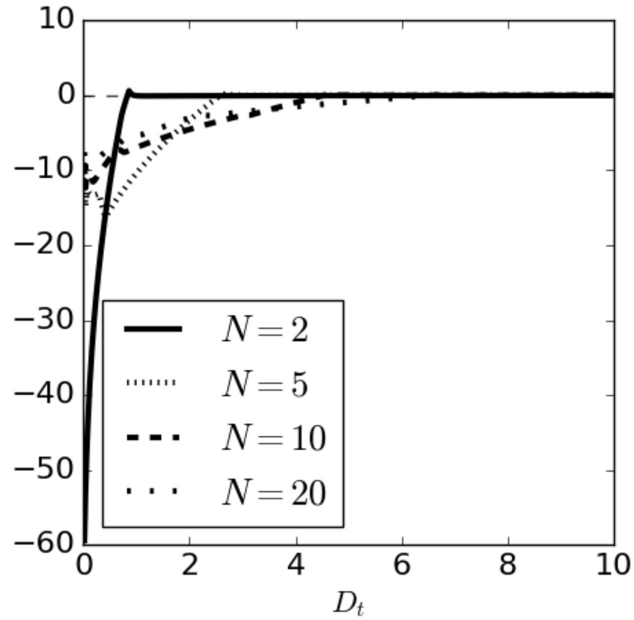


Fig. 9. Relative deviation of total leverage under margin constraint from complete market for 2, 5, 10, and 20 agents evenly distributed over $[1.1, 3.0]$.

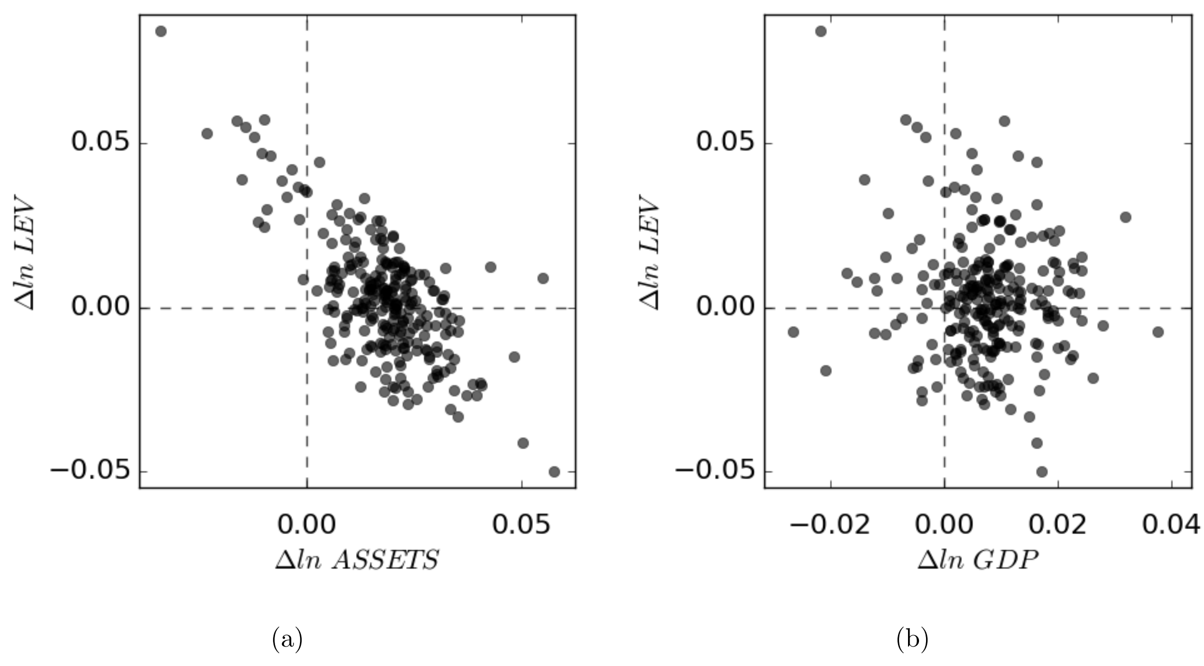
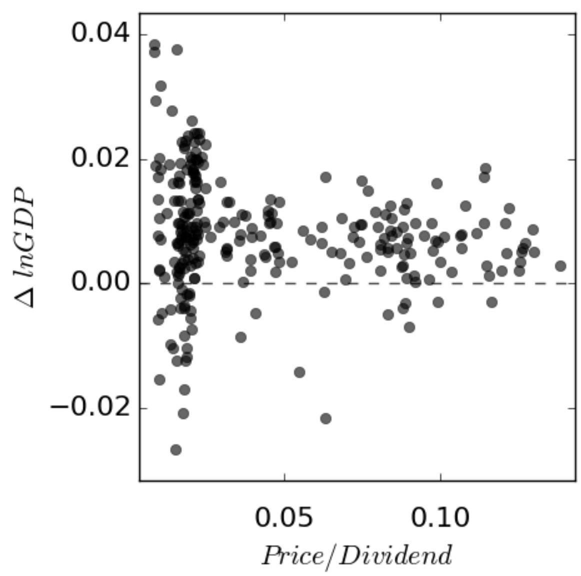
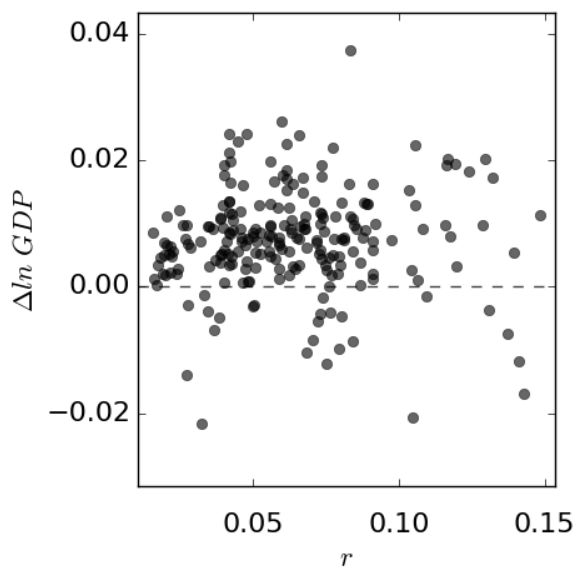


Fig. 10. Growth rate in leverage plotted against the growth rate in assets (Figure 10(a)) and against the growth rate in GDP (Figure 10(b)) for all sectors. Source: FRB Flow of Funds Data.

period, again from U.S. Flow of Funds data. The previously clear negative relationship has disappeared, implying the leverage cycle is ambiguous in this sense. However, this ambiguity may simply be that there exists some other explanatory variable which drives the cyclicality of leverage, in particular preference heterogeneity.



(a)



(b)

Fig. 11. Growth rate in GDP plotted against the price/dividend ratio (Figure 11(a)), proxied by the price of the S&P500 divided by GDP, and against the risk free rate (Figure 11(b)), proxied by the yield on constant maturity 10-year treasuries. Source: FRB Flow of Funds Data and FRED.

One proxy for preference heterogeneity is the price-dividend ratio. As we saw in section 2, asset prices will be high relative to dividends and vice-versa when the marginal agent in the economy is less risk averse. Figure 11(a) plots the growth rate in GDP against the price of the S&P 500 divided by GDP (a measure of the price/dividend ratio of the total economy). Indeed we see that there is substantial dispersion in this measure. The price/dividend ratio bunches towards the origin as asset prices have been rising over time, but there is little evidence for a clear positive or negative relationship with GDP growth. For this reason, we can consider the correlations between these variables, captured by the following regression:

$$\Delta ln\ Lev = \alpha + \beta_1 \Delta ln\ GDP + \beta_2 \Delta ln\ GDP * \frac{S}{D} + \beta_3 \frac{S}{D}$$

The cyclicality of the leverage cycle is then captured by the slope with respect to the growth

	Nonfinancial Corporations	Nonfinancial Private Business	HH's and Nonprofits	All Sectors
	(1)	(2)	(3)	(4)
Intercept	-0.0042	-0.0078	-0.0103	-0.0046
	(0.0618)	(0.0610)	(0.0599)	(0.0590)
$\Delta ln\ GDP$	0.1034	0.2117**	0.2827***	0.2397**
	(0.0994)	(0.0980)	(0.0963)	(0.0949)
S/D	-0.0045	-0.0119	-0.0209	-0.2107**
	(0.0874)	(0.0862)	(0.0846)	(0.0834)
$\Delta ln\ GDP * S/D$	-0.1727	-0.2726**	-0.3610***	-0.5157***
	(0.1070)	(0.1055)	(0.1036)	(0.1021)

Standard errors in parentheses.
$^{*}: p \leq 0.1$, $^{**}: p \leq 0.05$, $^{***}: p \leq 0.01$

Table 1: Regression results for dependent variable $\Delta ln\ Lev$ for different sectors of the economy. A positive and significant coefficient on $\Delta ln\ GDP$ implies procyclicality, while a negative and significant coefficient on the interaction with S/D implies counter-cyclicality when the price dividend ratio is high. Note: Variables are normalized using z-score.

rate in GDP, that is

$$\partial_{\Delta ln\ GDP} \Delta ln\ Lev = \beta_1 + \beta_2 \frac{S}{D}$$

The leverage cycle is pro- or counter-cyclical as this value is positive or negative, respectively Table 1 reports the results for several specifications, studying different subsamples of the economy.

The results imply that the cyclicality of leverage is not the same for all values of the price-dividend ratio. Column 4 gives results for all sectors included in the US Flow of Funds. Leverage growth is positively correlated with GDP growth when the price dividend ratio is low. As asset prices rise the effect changes sign and the correlation becomes negative. Changes in the price-dividend ratio imply changes in the preferences of the marginal agent pricing risky assets. When the price-dividend ratio is low the marginal agent is risk averse, while when the price-dividend ratio is high the marginal agent is more risk neutral. Thus the leverage cycle is pro-cyclical when risk-averse agents dominate and counter-cyclical when risk-neutral agents dominate.

This result is fairly robust to other measures of marginal preferences. One problem could be the heteroscedasticity exhibited by GDP growth over the price/dividend ratio in Figure 11(a). Consider the risk free rate as a proxy for the marginal agent, which is plotted in Figure 11(b) against GDP growth. In this case the dispersion of GDP growth is more

	Nonfinancial Corporations	Nonfinancial Private Business	HH's and Nonprofits	All Sectors
	(1)	(2)	(3)	(4)
Intercept	0.0170	0.0838	0.0948	0.0227
	(0.0674)	(0.0638)	(0.0674)	(0.0672)
$\Delta ln\ GDP$	−0.5144***	−0.4412**	−0.5330***	−0.9689***
	(0.1907)	(0.1808)	(0.1908)	(0.1904)
r	0.0164	−0.0847	0.0018	0.0549
	(0.0789)	(0.0748)	(0.0790)	(0.0788)
$\Delta ln\ GDP * r$	0.4121**	0.4095**	0.5131***	0.7603***
	(0.1700)	(0.1611)	(0.1701)	(0.1697)

Standard errors in parentheses.
$^{*}: p \leq 0.1$, $^{**}: p \leq 0.05$, $^{***}: p \leq 0.01$

Table 2: Regression results for dependent variable $\Delta ln\ Lev$ for different sectors of the economy. A negative and significant coefficient on $\Delta ln\ GDP$ implies counter-cyclicality, while a positive and significant coefficient on the interaction with r implies pro-cyclicality when the interest rate is high. As opposed to Table 1, r is high when the risk averse agent dominates, exactly when the price-dividend ratio is low. Note: Variables are normalized using z-score.

uniform over values of the interest rate. Define a similar set of regressions as before, i.e.:

$$\Delta ln\ Lev = \alpha + \beta_1 \Delta ln\ GDP + \beta_2 \Delta ln\ GDP * r + \beta_3 r$$

Again the cyclicality is captured by the slope with respect to the growth rate in GDP:

$$\partial_{\Delta ln\ GDP} \Delta ln\ Lev = \beta_1 + \beta_2 r$$

In this case we should expect the sign to flip. The interest rate is high when the marginal agent is risk-averse and low when the marginal agent is risk-neutral. Table 2 reports the results. Leverage growth co-moves positively with GDP growth and the interest rate is high and negatively when the interest rate is low. This result again implies that the cyclicality of the leverage cycle depends in the same way as before on the preferences of the marginal agent.

The regression results highlight how the cyclicality of the leverage cycle relates to financial variables and, in turn, preferences. Agents are likely to be constrained when asset prices are low, producing a pro-cyclical leverage cycle. Agents will be far from their constraint when asset prices are high, producing a counter-cyclical the leverage cycle. Asset price movements are explained by changes in the marginal agent in the economy, as seen in section 2.

5. Conclusion

In this paper I've shown how one can solve a model of preference heterogeneity when agents face convex portfolio constraints. The equilibrium and solution method are novel, to my knowledge. The methodological contribution goes beyond the present setting to any model with homogeneous (in the functional sense) utility functions and incomplete markets. In particular the method could be applied to macroeconomic models such as Krusell and Smith (1998) when preferences are of the right type. Future work on this topic should build incrementally, introducing a stochastic endowment and more general preferences. The economic contribution is to show how the degree of heterogeneity actually buffers some aspects of financial crises driven by margin constraints, but not others. In particular leverage cycles are less severe and credit contractions reduced when agents are more diverse, and financial variables deviate from complete markets to a lesser degree. In addition I've documented a new stylized fact predicted by the model, namely that leverage is both pro- and counter-cyclical depending on the level of aggregate consumption.

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Appendix A. Proofs

Proof of Proposition 1. Take the market clearing condition for consumption:

$$D_t = \frac{1}{N}\sum_j c_{jt} = \frac{1}{N}\sum_j \left(\frac{c_{jt}^{-\gamma_j}}{c_{it}^{-\gamma_i}}\right)^{\frac{-1}{\gamma_j}} \left(\frac{c_{it}}{D_t}\right)^{\frac{\gamma_i}{\gamma_j}} D_t^{\frac{\gamma_i}{\gamma_j}} \tag{20}$$

Substitute Eq. (5) and re-arrange to find Eq. (6). Define the wealth/consumption ratio of an individual as $V_i(Y) = X_{it}/c_{it}$ for some vector of state variables Y. Take the market clearing condition in wealth

$$S_t = \frac{1}{N}\sum_j X_{jt} = \frac{1}{N}\sum_j V_j(Y)c_{jt} = \frac{1}{N}\sum_j V_j(Y)\left(\frac{c_{jt}^{-\gamma_j}}{c_{it}^{-\gamma_i}}\right)^{\frac{-1}{\gamma_j}} \left(\frac{c_{it}}{D_t}\right)^{\frac{\gamma_i}{\gamma_j}} D_t^{\frac{\gamma_i}{\gamma_j}}$$

Again substitute Eq. (5) into the last equality and re-arrange to find

$$0 = \frac{1}{N}\sum_j V_j(Y)\left[\left(\frac{\Lambda_j H_{jt}}{\Lambda_i H_{it}}\right)^{-\frac{1}{\gamma_j}} \omega_{it}^{\frac{\gamma_i}{\gamma_j}} D_t^{\frac{\gamma_i}{\gamma_j}-1} - \omega_{jt}\right] \tag{21}$$

Notice that Eq. (6) and Eq. (21), conditional on knowledge of the wealth consumption ratios form a system of $2N$ equations in $2N$ unknowns. We choose $Y = D$ to be the minimal state vector, implying that this system of equations implies the ratios of marginal utilities and consumption weights for every value of D. This is not to say that this is the only equilibrium, but the remainder of the paper calculates this equilibrium and proves that it is an optimum. □

Proof of Proposition 2. Transform the variables in Eqs. (6) and (7) by a normalization, such that

$$\omega_{it} = \hat{\omega}_{it}\sum_i \omega_{it} \tag{22}$$

$$H_{it} = \hat{H}_{it}\sum_i H_{it} \tag{23}$$

These new variables combined in a vector live on the $2N$ standard simplex: $Z = (\hat{H}_{it}, \hat{\omega}_{it}) \in \Delta_{2N}$, which is a closed, convex set. Now substitute these variables into the equations in Eqs. (6) and (7) and define the resulting system as a vector value function $F(Z)$ such that $F(Z) = 0$. Finally, add Z to both sides and define a new function $G(\cdot)$ such that $G(Z) = (F(Z)+Z)/\|F(Z)+Z\|$. This implies that $G : \Delta_{2N} \to \Delta_{2N}$ satisfies the conditions of Brouwer's fixed point theorem and thus there exists at least one fixed point. □

Proof of Proposition 3. Take the market clearing condition in consumption and divide through by agent i's consumption

$$\frac{1}{N}\sum_j c_{jt} = D_t \Leftrightarrow c_{it} = \frac{c_{it}}{\frac{1}{N}\sum_j c_{jt}} D_t = \left(\frac{N\left(e^{\rho t}\Lambda_i H_{it}\right)^{\frac{-1}{\gamma_i}}}{\sum_j \left(e^{\rho t}\Lambda_j H_{jt}\right)^{\frac{-1}{\gamma_j}}} \right) D_t = \omega_{it} D_{it}$$

where ω_{it} represents an individual's consumption weight and is given by

$$\omega_{it} = \frac{N\left(e^{\rho t}\Lambda_i H_{it}\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^{N} \left(e^{\rho t}\Lambda_j H_{jt}\right)^{\frac{-1}{\gamma_j}}}$$

Assume individual consumption follows a GBM

$$\frac{dc_{it}}{c_{it}} = \mu_{cit} dt + \sigma_{cit} dW(t) \tag{24}$$

Apply Itô's lemma to Eq. (5) and solve for μ_{cit} and σ_{cit}

$$\mu_{cit} = \frac{r_t - \rho + \delta_i(\nu_{it})}{\gamma_i} + \frac{1+\gamma_i}{\gamma_i^2}\frac{1}{2}\left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right)^2 \quad , \quad \sigma_{c^i}(t) = \frac{1}{\gamma_i}\left(\theta + \frac{\nu_{it}}{\sigma_t}\right)$$

Apply Itô's lemma to the market clearing condition for consumption and match coefficients to find

$$\mu_D = \frac{1}{N}\sum_{i=1}^{N} \omega_{it}\mu_{cit} \quad , \quad \sigma_D = \frac{1}{N}\sum_{i=1}^{N} \omega_{it}\sigma_{cit}$$

Now substitute the values for consumption drift and diffusion and solve for the interest rate and the market price of risk:

$$\theta_t = \frac{N}{\sum_i \frac{\omega_{it}}{\gamma_i}} \left(\sigma_D - \frac{1}{\sigma_t N}\sum_i \frac{\omega_{it}\nu_{it}}{\gamma_i} \right)$$

$$r_t = \frac{N}{\sum_i \frac{\omega_{it}}{\gamma_i}} \left(\mu_D + \frac{\rho}{N}\sum_i \frac{\omega_{it}}{\gamma_i} - \frac{1}{N}\sum_i \frac{\omega_{it}}{\gamma_i}\delta_i(\nu_{it}) - \frac{1}{2N}\sum_i \frac{1+\gamma_i}{\gamma_i}\left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right)^2 \omega_{it} \right)$$

□

Proof of Proposition 4. Apply Itô's lemma to $\omega_{it} = \frac{c_{it}}{D_t}$ and match coefficients to find the dynamics of consumption weights in Eq. (12) and Eq. (13). □

Proof of Propostions 5 and 6. Assume there exists a Markovian equilibrium in D_t. Then an individual's Hamilton-Jacobi-Bellman (HJB) equation writes

$$\begin{aligned} 0 = \max_{c_{it},\pi_{it}} \Bigg\{ & e^{-\rho t}\frac{c_{it}^{1-\gamma_i}}{1-\gamma_i} + \frac{\partial J_{it}}{\partial t} + \left[X_{it}\left(r_t + \delta_i(\nu_{it}) + \pi_{it}\sigma_t\left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right)\right) - c_{it}\right]\frac{\partial J_{it}}{\partial X_{it}} \\ & + \mu_D D_t \frac{\partial J_{it}}{\partial D_t} + \sigma_D\sigma_t\pi_{it}D_tX_{it}\frac{\partial^2 J_{it}}{\partial X_{it}\partial D_t} + \frac{1}{2}\left[X_{it}^2\pi_{.it}^2\sigma_t^2\frac{\partial^2 J_{it}}{\partial X_{it}^2} + \sigma_D^2 D_t^2\frac{\partial^2 J_{it}}{\partial D_t^2}\right]\Bigg\} \end{aligned} \tag{25}$$

subject to the transversality condition $\mathbb{E}_t J_{it} \to 0$ for all i s.t. $\gamma_i > \underline{\gamma}$, as the agent with the lowest risk aversion will dominate in the long run (Cvitanić et al. (2011)). First order conditions imply

$$c_{it} = \left(e^{\rho t}\frac{\partial J_{it}}{\partial X_{it}}\right)^{\frac{-1}{\gamma_i}} \tag{26}$$

$$\pi_{it} = -\left(X_{it}\sigma_t\frac{\partial^2 J_{it}}{\partial X_{it}^2}\right)^{-1}\left[\left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right)\frac{\partial J_{it}}{\partial X_{it}} + \sigma_D D_t\frac{\partial^2 J_{it}}{\partial X_{it}\partial D_t}\right] \tag{27}$$

Assume that the value function is separable as

$$J_{it}(X_{it}, D_t) = e^{-\rho t}\frac{X_{it}^{1-\gamma_i}V_i(D)^{\gamma_i}}{1-\gamma_i} \tag{28}$$

Substituting Eq. (28) into Eqs. (26) and (27) gives

$$c_{it} = \frac{X_{it}}{V_i(D)} \tag{29}$$

$$\pi_{it} = \frac{1}{\gamma_i\sigma_t}\left(\gamma_i\sigma_D D_t\frac{V_i'(D)}{V_i(D)} + \theta_t + \frac{\nu_{it}}{\sigma_t}\right) \tag{30}$$

which shows that $V_i(D)$ is the wealth-consumption ratio as a function of the dividend. Next, substitute Eqs. (28) to (30) into Eq. (25) and simplify to find

$$\begin{aligned} 0 = & 1 + \frac{\sigma_D^2 D^2}{2}V_i''(D) + \left[\frac{1-\gamma_i}{\gamma_i}\left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right)\sigma_D + \mu_D\right]DV_i'(D) \\ & + \left[(1-\gamma_i)(r_t + \delta_i(\nu_{it})) - \rho + \frac{1-\gamma_i}{2\gamma_i}\left(\theta_t + \frac{\nu_{it}}{\sigma_t}\right)^2\right]\frac{V_i(D)}{\gamma_i} \end{aligned} \tag{31}$$

which gives an ode for the wealth-consumption ratio over the state space. The boundary conditions are given by recognizing that the limit in $D \to \infty$ is an economy in autarky dominated by the most risk neutral agent (Cvitanić et al. (2011); Chabakauri (2015)). Similarly, as $D \to 0$ the most risk averse agent dominates. □

Proof of Proposition 7. Define the price-dividend ratio as a function of the single state variable: $\mathcal{S}(D_t) = \frac{S_t}{D_t}$. Apply Itô's lemma to $D_t\mathcal{S} = S_t$ and match coefficients to find

$$\begin{aligned}
\mu_t &= D_t^2 + \frac{(\sigma_D D_t)^2}{2}\frac{\mathcal{S}''(D_t)}{\mathcal{S}(D_t)}D_t + D_t\mu_D + \frac{\mathcal{S}'(D_t)}{\mathcal{S}(D_t)}(\sigma_D D_t)^2 \\
\sigma_t &= \sigma_D\left(1 + D_t\frac{\mathcal{S}'(D_t)}{\mathcal{S}(D_t)}\right)
\end{aligned}$$

Taking the market clearing condition for wealth, rewrite $\mathcal{S}(D_t)$ as a function of D_t:

$$S_t = \frac{1}{N}\sum_i X_{it} \Leftrightarrow \frac{S_t}{D_t} = \mathcal{S}(D_t) = \frac{1}{N}\sum_i \frac{X_{it}}{D_t} = \frac{1}{N}\sum_i \frac{X_{it}}{c_{it}}\frac{c_{it}}{D_t} = \frac{1}{N}\sum_i V(D_t)\omega_{it}$$

which gives $\mathcal{S}(D_t)$ given that $\omega_{it} = \omega_i(D_t)$ □

Proof of Proposition 8. For a homogeneous margin constraint, $\nu_{it} \leq 0$ and $m \geq 0$, thus $\nu_{it}m \leq 0$ (Cvitanić and Karatzas (1992); Chabakauri (2015)). Additionally, $\pi_{it} \leq m$. Substituting the solution for π_{it} from Eq. (30) into the latter inequality and recognizing that, by the Kuhn-Tucker conditions at least one of the inequalities holds with equality gives the result. □

Proof of Proposition 9. This proof proceeds identically to Chabakauri (2015). Let $V_i(D) \in C^1[0,\infty) \cup C^2[0,\infty)$, $0 < V_i(D) \leq C_1$, $|\pi_{it}\sigma_t| < C1$, and $|\theta_t + \nu_{it}/\sigma_t| < C_1$, where C_1 is a constant. Additionally, assume

$$\mathbb{E}\int_0^\infty e^{-\rho t}\frac{c_{it}^{1-\gamma_i}}{1-\gamma_i}dt < \infty \tag{32}$$

$$\mathbb{E}\int_0^T J_i(X_{it}, D_t, t)^2 dt < \infty \qquad \forall\ T > 0 \tag{33}$$

$$\limsup_{T\to\infty} \mathbb{E}J_i(X_{it}, D_t, t) \geq 0 \tag{34}$$

Define $U_t = \int_0^t e^{-\rho t}c_{i\tau}^{1-\gamma_i}/(1-\gamma_i)d\tau + J_i(X_{it}, D_t, t)$, which satisfies $dU_t = \mu_{Ut}dt + \sigma_{Ut}dW_t$ such that

$$\mu_{Ut} = \left(e^{-\rho t} \frac{c_{it}^{1-\gamma_i} - 1}{1-\gamma_i} + \frac{\partial J_{it}}{\partial t} + \left[X_{it} \left(r_t + \delta(\nu_{it}) + \pi_{it}\sigma_t \left(\theta_t + \frac{\nu_{it}}{\sigma_t} \right) \right) - c_{it} \right] \frac{\partial J_{it}}{\partial X_{it}} \right.$$
$$\left. + \mu_D D_t \frac{\partial J_{it}}{\partial D_t} + \sigma_D \sigma_t \pi_{it} D_t X_{it} \frac{\partial^2 J_{it}}{\partial X_{it} \partial D_t} + \frac{1}{2} \left[X_{it}^2 \pi_{.it}^2 \sigma_t^2 \frac{\partial^2 J_{it}}{\partial X_{it}^2} + \sigma_D^2 D_t^2 \frac{\partial^2 J_{it}}{\partial D_t^2} \right] \right)$$
$$- (\nu_{it}\pi_{it} + \delta(\nu_{it})) X_{it} \frac{\partial J_{it}}{\partial X_{it}} \tag{35}$$
$$\sigma_{Ut} = J_{it} \left((1-\gamma_i)\pi_{it}\sigma_t + \gamma_i D_t \sigma_D \frac{V_i'(D)}{V_i(D)} \right) = J_{it} \left(\pi_{it}\sigma_t - \theta_t - \frac{\nu_{it}}{\sigma_t} \right) \tag{36}$$

The first term in μ_{Ut} is simply the PDE inside the max operator in Eq. (25), and is thus weakly negative. The second term is as well, as $\nu_{it}\pi_{it} + \delta(\nu_{it}) \geq 0$ by definition and $\partial_{X_{it}} J_{it} \geq 0$. Thus $\mu_{Ut} \leq 0$. By the boundedness conditions, U_t is integrable and because its drift is negative it is a supermartingale, thus $U_t \geq \mathbb{E}_t U_T \ \forall \ t \leq T$, which is equivalent to

$$J_i(X_{it}, D_t, t) \geq \mathbb{E}_t \int_t^T e^{-\rho(\tau - t)} \frac{c_{i\tau}^{1-\gamma_t}}{1-\gamma_i} d\tau + \mathbb{E}_t J_i(X_{it}, D_t, T) \tag{37}$$

Since the first term is monotonic in T, by Eq. (34) and by the monotone convergence theorem we have

$$J_i(X_{it}, D_t, t) \geq \mathbb{E}_t \int_t^\infty e^{-\rho(\tau - t)} \frac{c_{i\tau}^{1-\gamma_t}}{1-\gamma_i} d\tau \tag{38}$$

Now to show the opposite, we first show that $\mathbb{E}_t J_i(X_{i\tau}, D_\tau, \tau) \to 0$ as $\tau \to \infty$. Applying Itô's lemma to $J_i(X_{it}, D_t, t)$ and following similar steps as before, we find $dJ_{it} = J_{it}[\mu_{Jt} dt + \sigma_{Jt} dW_t]$ where

$$\mu_{Jt} = \frac{-1}{V_i(D)}$$
$$\sigma_{Jt} = \pi_{it}\sigma_t - \theta_t - \frac{\nu_{it}}{\sigma_t}$$

by the first order conditions Eqs. (29) and (30). By the boundedness assumptions σ_{Jt} satisfies Novikov's conditions and we have that $d\eta_t = \eta_t \sigma_{Jt} dW_t$ acts as a change of measure to remove the Brownian term in J_{it}. We have

$$|\mathbb{E}_t J_i(X_{i\tau}^*, D_\tau, \tau)| \leq \mathbb{E}_t \left[|J_{i\tau}| exp \left\{ - \int_t^T \frac{1}{V_i(D)} du \right\} \frac{\eta_\tau}{\eta_t} \right]$$
$$\leq |J_{it}| e^{-(T-t)/C_1} \mathbb{E}_t \frac{\eta_\tau}{\eta_t} = |J_{it}| e^{-(T-t)/C_1}$$

Taking the limit in T gives the result.

Finally, define U_t^* as for U_t, except evaluated at the optimum consumption. Then

$$dU_t^* = J_{it}\left(\pi_{it}\sigma_t - \theta_t - \frac{\nu_{it}}{\sigma_t}\right)dW_t \tag{39}$$

Again applying Novikov's condition we get that U_t^* is an exponential martingale, which gives (after integrating Eq. (39))

$$J_i(X_{it}, D_t, t) = \mathbb{E}_t \int_t^T e^{-\rho(\tau-t)} \frac{(c_{it}^*)^{1-\gamma_t}}{1-\gamma_i} d\tau + \mathbb{E}_t J_i(X_{it}^*, D_t, T)$$

Finally, by the intermediate result the last term goes to zero, showing that we do indeed have an optimum. □

Appendix B. Numerical Method

The problem presented by the equilibrium under margin constraints has several difficult features which make it challenging from a numerical perspective. First, the system of ODE's in Eq. (14) represents a highly non-linear system, as the coefficients depend in a non-trivial way on the solution itself. Second, Eq. (6) represents a set of constraints on the solution to the ODE's. These two facts combined place the problem under the framework of "Non-Linear Differential Algebraic Systems". Finally, at the point where an individual's portfolio constraint binds there will exist a singularity. The consumption weights will be kinked (and thus not differentiable) creating a jump in the coefficients. This phase transition creates a free boundary problem, as it is impossible to determine analytically the point at which the constraints bind. All of these points combined make this problem particularly challenging (for more on the mathematical particularities of these topics see Wanner and Hairer (1991); Hirsch et al. (2012); Friedman (1982)).

Luckily, the numerical solution exhibits characteristics which make an ad-hoc solution algorithm possible. First, the equations seem to be hyperbolic (LeVeque (2002)), implying that an implicit-explicit backwards Euler approach requires only the terminal condition and avoids noise introduced by the Dirichlet boundary conditions in Eq. (15). Second, the structure of the model is very similar to that of a Mean-Field Game. Treating the differential equations and constraints iteratively, as in Achdou et al. (2014) proves to be stable. Lastly, although there exists a singularity, one can reformulate the definition of volatility to avoid numerically approximating the derivative across the free boundary. Using this and the fact that the solution to the ODE's are smooth, a classical finite difference method suffices and one can avoid the use of viscosity solutions.

B.1. A More Precise Volatility

To deal with the volatility being non-differentiable, consider the definition of the price/dividend ratio, $\mathcal{S}_t = S_t/D_t$. If we assume the price dividend ratio has the dynamics $d\mathcal{S}_t = \mu_{*t}dt + \sigma_{*t}dW_t$, we can show that

$$\begin{aligned}\mu_t &= \mu_{*t} + \mu_D + \sigma_D\sigma_{*t}\\ \sigma_t &= \sigma_{*t} + \sigma_D\end{aligned}$$

Next, consider the definition of the price/dividend ratio given in Eq. (18) and apply Itô's lemma. Matching coefficients gives

$$\begin{aligned}\mu_{*t} &= \frac{1}{N}\sum_i \omega_{it}\left[V_i(D_t)\mu_{\omega it}V_i'(D_t)(\mu_D + \sigma_{\omega it}\sigma_D)D_t + \frac{\sigma_D^2 D_t^2}{2}V_i''(D_t)\right]/\mathcal{S}(D_t)\\ \sigma_{*t} &= \frac{1}{N}\sum_i \omega_{it}\left[V_i(D_t)\sigma_{\omega it} + V_i'(D_t)\sigma_D D_t\right]/\mathcal{S}(D_t)\end{aligned}$$

Combining the above expressions gives volatility, but with derivatives across only the wealth consumption ratio, which is a smooth function:

$$\sigma_t = \sigma_D + \frac{1}{N}\sum_i \omega_{it}\left[V_i(D_t)\sigma_{\omega it} + V_i'(D_t)\sigma_D D_t\right]/\mathcal{S}(D_t)$$

where $\sigma_{\omega it}$ is given in Proposition 4.

B.2. DAE Solution

To solve the system of constrained ODE's I use a finite difference approach combined with a Picard type iteration. Picard iteration is necessary because of the non-linear aspect of the differential equation. That is, if we define the vector of wealth/consumption ratios as $\overrightarrow{V}$, the associated gradient as $\nabla\overrightarrow{V}$, and the vector of undifferentiated endogenous variables $Z_i = (\omega_{it}, H_{it}, \nu_{it})$ as $\overrightarrow{Z}$, then we can write the DAE system as

$$0 = a_i\left(D_t, \overrightarrow{V}, \nabla\overrightarrow{V}, \overrightarrow{Z}\right)V_i''(D_t) + b_i\left(D_t, \overrightarrow{V}, \nabla\overrightarrow{V}, \overrightarrow{Z}\right)V_i'(D_t) + c(D_t)V_i(D_t) + 1 \qquad \forall\ i \tag{40}$$

$$0 = F\left(D_t, \overrightarrow{V}, \nabla\overrightarrow{V}, \overrightarrow{Z}\right) \tag{41}$$

where $F(\cdot,\cdot,\cdot,\cdot)$ represents the market clearing conditions given in Proposition 1. To solve this problem, I use an iterative solution approach. First, taking a solution for $\overrightarrow{Z}$ as given, I solve the ODE's via an implicit Picard iteration until convergence. Given this solution for $\overrightarrow{V}$, I solve the market clearing conditions for $\overrightarrow{Z}$.

A description of the algorithm goes as follows: Discretize the state space in D_t into K evenly spaced points, truncated the semi-infinite interval. Assume an initial guess for $\overrightarrow{V}^k_{0,0} = \Delta_t$ for each point in the state space. Iterate over the following:

- Given the current guess $\overrightarrow{V}^k_{t,m-1}$, solve Eq. (41) for a new guess for $\overrightarrow{Z}^k_{t,m}$.
- Test for convergence using the infinity norm:

$$error_Z = \max_k \left[|\overrightarrow{Z}^k_{t,m} - \overrightarrow{Z}^k_{t,m-1}| \right]$$

- If convergence, stop, else iterate over the following:
  - The system of ODE's takes on a finite difference approximation as

$$\begin{aligned} \frac{V^{k-1}_{i,t,m} - V^{k-1}_{i,t-1,m}}{\Delta_t} &+ a_i\left(D^k, \overrightarrow{V}^k_{t,m}, \frac{\overrightarrow{V}^{k-1}_{t,m} - \overrightarrow{V}^{k+1}_{t,m}}{2\Delta_D}, \overrightarrow{Z}^k_{t,m}\right) \frac{V^{k-1}_{i,t-1,m} - 2V^k_{i,t-1,m} + V^{k+1}_{i,t-1,m}}{\Delta_D^2} \\ &+ b_i\left(D^k, \overrightarrow{V}^k_{t,m}, \frac{\overrightarrow{V}^{k-1}_{t,m} - \overrightarrow{V}^{k+1}_{t,m}}{2\Delta_D}, \overrightarrow{Z}^k_{t,m}\right) \frac{V^{k-1}_{i,t-1,m} - V^{k+1}_{i,t-1,m}}{2\Delta_D} \\ &+ c(D^k)V^k_{i,t-1,m} = 0 \end{aligned}$$

    Notice this is linear in the unknown quantity $\overrightarrow{V}^k_{t-1,m}$. Solve for this quantity.
  - Iterate until two consecutive guesses are sufficiently close, using the infinity norm:

$$error_V = \max_k \left[|\overrightarrow{V}^k_{t,m} - \overrightarrow{V}^k_{t-1,m}| \right]$$

  - If convergence, return to outer loop, else continue.