UNIL | Université de Lausanne
Département de finance

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

FINANCE RESEARCH SEMINAR SUPPORTED BY UNIGESTION

"A Model of Safe Asset Determination"

Prof. Arvind KRISHNAMURTHY

Stanford University, Graduate School of Business

Abstract

What makes an asset a "safe asset"? We study a model where two countries each issue sovereign bonds to satisfy investors' safe asset demands. The countries differ in the float of their bonds and their resources/fundamentals available to rollover debts. A sovereign's debt is more likely to be safe if its fundamentals are strong relative to other possible safe assets, but not necessarily strong on an absolute basis. Debt float can enhance or detract from safety: If global demand for safe assets is high, a large float can enhance safety. The large float offers greater liquidity which increases demand for the large debt and thus reduces rollover risk. If demand for safe assets is low, then large debt size is a negative as rollover risk looms large. When global demand is high, countries may make fiscal/debt-structuring decisions to enhance their safe asset status. These actions have a tournament feature, and are self-defeating: countries may over-expand debt size to win the tournament. Coordination can generate benefits. The model sheds light on the effects of "Eurobonds" – i.e. a coordinated Euro-area-wide safe bond design. Eurobonds deliver welfare benefits only when they make up a sufficiently large fraction of countries' debts. Small steps towards Eurobonds may hurt countries and not deliver welfare benefits.

Wednesday, October 26, 2016, 10:30-12:00

Room 125, Extranef building at the University of Lausanne

A Model of Safe Asset Determination*

Zhiguo He† Arvind Krishnamurthy‡ Konstantin Milbradt§

February 12, 2016

Abstract

What makes an asset a "safe asset"? We study a model where two countries each issue sovereign bonds to satisfy investors' safe asset demands. The countries differ in the float of their bonds and their resources/fundamentals available to rollover debts. A sovereign's debt is more likely to be safe if its fundamentals are strong relative to other possible safe assets, but not necessarily strong on an absolute basis. Debt float can enhance or detract from safety: If global demand for safe assets is high, a large float can enhance safety. The large float offers greater liquidity which increases demand for the large debt and thus reduces rollover risk. If demand for safe assets is low, then large debt size is a negative as rollover risk looms large. When global demand is high, countries may make fiscal/debt-structuring decisions to enhance their safe asset status. These actions have a tournament feature, and are self-defeating: countries may over-expand debt size to win the tournament. Coordination can generate benefits. The model sheds light on the effects of "Eurobonds" – i.e. a coordinated Euro-area-wide safe bond design. Eurobonds deliver welfare benefits only when they make up a sufficiently large fraction of countries' debts. Small steps towards Eurobonds may hurt countries and not deliver welfare benefits.

*We thank Eduardo Davila, Emmanuel Farhi, Itay Goldstein, Christian Hellwig, Pierre-Olivier Gourinchas, Alessandro Pavan, Andrei Shleifer, Mathieu Taschereau-Dumouchel, Kathy Yuan, and conference/seminar participants at Columbia University, Harvard University, NBER IFM Summer Institute 2015, Northwestern University, SF Federal Reserve, Stanford University, UC-Berkeley, UC-Davis, USC, Amsterdam University Workshop for Safe Assets, AFA 2016, German Economist Abroad 2015, Imperial College London for helpful comments. This paper was previously circulated under the title, "A model of the reserve asset."

†University of Chicago, Booth School of Business, and NBER. Email: zhiguo.he@chicagobooth.edu.

‡Stanford University, Graduate School of Business, and NBER. Email: a-krishnamurthy@stanford.edu.

§Northwestern University, Kellogg School of Management, and NBER. Email: milbradt@northwestern.edu.

1 Introduction

US government debt is the premier example of a global safe asset. Investors around the world looking for a safe store of value, such as central banks, tilt their portfolios heavily towards US government debt. German government debt occupies a similar position as the safe asset within Europe. US and German debt appear to have high valuations relative to the debt of other countries with similar fundamentals, measured in terms of debt or deficit to income ratios. Moreover, as fundamentals in the US and Germany have deteriorated, these high valuations have persisted. Finally, as evident in the financial crises over the last five years, during times of turmoil, the value of these countries' bonds rise relative to the value of other countries' bonds in a flight-to-quality.

What makes US or German government debt a "safe asset"? This paper develops a model that helps understand the characteristics of an asset that make it safe, as well why safe assets display the phenomena described above. We study a model with many investors and two countries, each of which issues government bonds. The investors have a pool of savings to invest in the government bonds. Thus the bonds of one, or possibly both of the countries, will hold these savings and serve as a store of value. However, the debts are subject to rollover risk. The countries differ in their fundamentals, which measure their ability to service their debt and factor into their rollover risk; and debt sizes, which proxy for the financial depth or liquidity of the country's debt market. Our model links fundamentals and debt size to the valuation and equilibrium determination of asset safety.

In the model, an investor's valuation of a bond depends on the number of other investors who purchase that bond. If only a few investors demand a country's bond, the debt is not rolled over and the country defaults on the bond. For a country's bonds to be safe, the number of investors who invest in the bond must exceed a threshold, which is decreasing in the country's fundamentals (e.g., the fiscal surplus) and increasing in the size of the debt. The modeling of rollover risk is similar to Calvo [9] and Cole and Kehoe [12]. Investor actions are *complements* – as more investors invest in a country's bonds, other investors are incentivized to follow suit. Our perspective on asset safety emphasizes coordination, as opposed to (exclusively) the income process backing the asset, as in conventional analyses of credit risk. In the world, the assets that investors own as their safe assets are largely government debt, money and bank debt. For these assets, valuation has a significant

coordination component as in our model, underscoring the relevance of our perspective.

Besides the above strategic complementarity, the model also features strategic substitutability, as is common in models of competitive financial markets. Once the number of investors who invest in the bonds exceeds the threshold required to roll over debts, then investor actions become *substitutes*. Beyond the threshold, more demand for the bond that is in fixed supply drives up the bond price, leading to lower returns. Our model links the debt size to this strategic substitutability: for the same investor demand, a smaller debt size leads to a smaller return to investors.

The model predicts that relative fundamentals more so than absolute fundamentals are an important component of asset safety. Relative fundamentals matter because of the coordination aspect of valuation. Investors expect that other investors will invest in the country with better fundamentals, and thus relative valuation determines which country's bonds have less rollover risk and thus safety. This prediction helps understand the observations we have made regarding the valuation of US debt in a time of deteriorating fiscal fundamentals. In short, all countries' fiscal conditions have deteriorated along with the US, so that US debt has maintained and perhaps strengthened its safe asset status. The same logic can be used to understand the value of the German Bund (as a safe asset within the Euro area) despite deteriorating German fiscal conditions. The Bund has retained/enhanced its value because of the deteriorating fiscal conditions of other Euro area countries.

We further show that this logic can endogenously generate the negative β of a safe asset; that is, the phenomenon that safe asset values rise during a flight to quality. Starting from a case where the characteristics of one country's debt are so good that it is almost surely safe; a decline in world absolute fundamentals further reinforces the safe asset status of that country's debt, leading to an increase of its value. We can thus explain the flight-to-quality pattern in US government debt.

The model also predicts that debt size is an important determinant of safety. If the global demand for safe assets is high, then large debt size enhances safety. Consider an extreme example with a large debt country and a small debt country. If investors coordinate all of their investment into this small debt country, then the return on their investments will be small. That is the quantity of world demand concentrating on a small float of bonds will drive bond prices up to a point that investors' incentives in equilibrium will be to coordinate investment in the large debt. On the other hand, if global demand for safe assets is low, then investors will be concerned that the large debt

may not attract sufficient demand to rollover the debt. In this case, investors will tend to coordinate on the small debt size as the safe asset.

Our model offers some guidance on when the US government may lose its dominance as a provider of the world safe asset. Many academics have argued that we are and have been in a global savings glut, which in the model corresponds to a high global demand for safe assets. In this case, US government debt is likely to continue to be the safe asset unless US fiscal fundamentals deteriorate significantly relative to other countries, or if another sovereign debt can compete with the US government debt in terms of size. Eurobonds seem like the only possibility of the latter, although there is considerable uncertainty whether such bonds will exist and will have better fundamentals than the US debt. However, if the savings glut ends and the world moves to a low demand for safe assets, then our model predicts that US debt may become unsafe. In this case, investors may shift safe asset demand to an alternative high fundamentals country with a relatively low supply of debt, such as the German Bund.

We use our model to investigate the benefits of creating "Eurobonds." We are motivated by recent Eurobond proposals (see Claessens et al. [11] for a review of various proposals). A shared feature of the many proposals is to create a common Euro-area-wide safe asset. Each country receives proceeds from the issuance of the "common bond" which is meant to serve as the safe asset, in addition to proceeds from the sale of an individual country-specific bond. By issuing a common Eurobond, all countries benefit from investors' need for a safe asset, as opposed to just one country (Germany) which is the de-facto safe asset in the absence of a coordinated security design. As our model features endogenous determination of the safe asset, it is well-suited to analyze these proposals formally. Suppose that countries issue α share of common bonds and $1 - \alpha$ share as individual bonds. We ask, how does varying α affect welfare, and the probability of safety for each country? Our main finding is that welfare is only unambiguously increased for α above a certain threshold. Above this threshold, the common-bond structure enhances the safety of both common bonds and individual bonds. Below the threshold, however, welfare can be increasing or decreasing, depending on the assumed equilibrium; and one country may be made worse off while another may be made better off by increasing α. We conclude that a successful Eurobond proposal requires a significant amount of coordination and volume / size of said Eurobonds.

We also use our model to study incentives to change debt size, when doing so may enhance

safe asset status. We study a case where two countries have a "natural" debt size, determined for example by their GDP, but can deviate from its natural debt size by some adjustment cost. Two interesting cases emerge. When countries are roughly symmetric – similar natural debt size – and when global demand for safe assets is high, countries will engage in a rat race to become the safe asset. Starting from the natural debt sizes, and holding fixed the size decision of one country, the other country will have an incentive to increase its debt size since the larger debt size can confer safe asset status. But then the first country will have an incentive to respond in a similar way, and so on so forth. In equilibrium, both countries will expand in a self-defeating manner beyond their natural debt size. This prediction of the model can help to shed some light on the expansion of relatively safe stocks of debt in the US (GSE debt) and Europe (sovereign debt) in the build-up to the crisis. These expansions have ultimately ended badly. The model identifies a second case, when countries are asymmetric and one country is the natural "top dog." In this case, the larger debt country will have an incentive to reduce debts to the point that balances rollover risk and retaining safe asset status, while the smaller country will have an incentive to expand its debt size. The model is suggestive that asymmetry leads to better outcomes than symmetry.

Literature review. There is a literature in international finance on the reserve currency through history. Historians identify the UK Sterling as the reserve currency in the pre-World War 1 period, and the US Dollar as the reserve currency post-World War 2. There is some disagreement about the interwar period, with some scholars arguing that there was a joint reserve currency in this period. Eichengreen [16, 17, 18] discusses this history. Gourinchas et al. [22] present a model of the special "exorbitant privilege" role of the US dollar in the international financial system. A reserve currency fulfills three roles: an international store of value, a unit of account, and a medium of exchange (Krugman [34], Frankel [19]). Our paper concerns the store of value role. There is a broader literature in monetary economics on the different roles of money (e.g., Kiyotaki and Wright [31], Banerjee and Maskin [3], Lagos [35], Freeman and Tabellini [20], Doepke and Schneider [15]), and our analysis is most related to the branch of the literature motiving money as a store of value. Samuelson [41] presents an overlapping generation model where money serves as a store of value, allowing for intergenerational trade. Diamond [14] presents a related model but where government debt satisfies the store of value role. In this class of models, there is a need for a store of value,

but the models do not offer guidance on which asset will be the store of value. For example, it is money in Samuelson [41] and government debt in Diamond [14]. In our model, the store of value determination is endogenous.

Our paper also belongs to a growing literature on safe asset shortages. Theoretical work in this area explores the macroeconomic and asset pricing implications of such shortages (Holmstrom and Tirole [30], Caballero et al. [8], Caballero and Krishnamurthy [6], Maggiori [36], Caballero and Farhi [7]). There is also an empirical literature documenting safe asset shortages and their consequences (Krishnamurthy and Vissing-Jorgensen [32, 33], Greenwood and Vayanos [23], Bernanke et al. [5]). We presume that there is a macroeconomic shortage of safe assets, and our model endogenously determines the characteristics of government debt supply that satisfies the safe asset demand.

The element of rollover risk in our model is in the spirit of Calvo [9] and Cole and Kehoe [12]. Rollover risk is also an active research area in corporate finance, with prominent contributions by Diamond [13], and more recently, Morris and Shin [39], He and Xiong [28, 27], and He and Milbradt [25, 26]. We utilize global games techniques (Carlsson and van Damme [10]; Morris and Shin [37]; and others) to link countries' fundamentals to the determination of asset safety. In our economy agent actions can be strategic complements, as in much of this literature, but different from the literature (e.g., Rochet and Vives [40]) can also be strategic substitutes. In this sense, our paper is related to Goldstein and Pauzner [21], who derive the unique equilibrium in a bank-run model with strategic substitution effects. The strategic substitution effect in our model is however stronger than Goldstein and Pauzner [21] and can lead to multiple equilibria, similar to Angeletos et al. [1, 2]. In our analysis, when these strategic substitution effects are sufficiently strong, we construct an equilibrium in which investor strategies are non-monotone. This equilibrium is new and a contribution to the global games literature. We label this equilibrium, which closely resembles a mixed-strategy equilibrium, an "oscillating" equilibrium. Last, a simplified version of the current model with an assumed equilibrium selection rule instead of global game techniques is given in He et al. [29].

In our model, debt size confers greater liquidity in the sense that a small buy/sell has a smaller price impact. In the search literature, papers such as Vayanos and Weill [42] show that a larger float of debt can result in greater liquidity. This occurs because it is easier to finding trading partners when float is larger. Thus, liquidity has a coordination element via ease of trading that is enhanced

by float. In our model, the coordination element is through rollover risk, which interacts with debt float/liquidity. Note that the Vayanos and Weill [42] analysis could as well apply to risky assets as to safe assets. We are centrally interested in describing safe assets, which is why we study rollover risk and the feedback of liquidity into safety through rollover risk.[1]

2 Model

2.1 The Setting

Consider a two-period model with two countries, indexed by i, and a continuum of homogeneous risk-neutral investors, indexed by j. At date 0 each investor is endowed with one unit of consumption good, which is the numeraire in this economy. Investors invest in the bonds offered by these two countries to maximize their expected date 1 consumption, and there is no other storage technology available. This latter restriction is important to the analysis as will be clear, but can be weakened as we describe in Section 3.4.

There is a large country, called country 1, and a small country, called country 2. We normalize the debt size of the large country to be one (i.e., $s_1 = 1$), and denote the debt size of the small country by $s \equiv s_2 \in (0, 1]$. Each country sells bonds at date 0 promising repayment at date 1. The size determines the total face value (in terms of promised repayment) of bonds that each country sells: the large (small) country offers 1 (s) units of sovereign bonds. Hence the aggregate bond supply is $1 + s$. All bonds are zero coupon bonds. We can think of the large country as the US and the small country as Canada.

The aggregate measure of investors, which is also the aggregate demand for bonds, is $1 + f$, where $f > 0$ is a constant parameterizing the aggregate savings need. To save, we assume that investors place *market orders* to purchase sovereign bonds. In particular, since purchases are via *market orders*, the aggregate investor demand does not depend on the equilibrium price.[2] Denote by p_i the equilibrium price of the bond issued by country i. Since there is no storage technology

[1] Our paper complements the neoclassical asset pricing literature explaining differences in cross-country currency returns based on country size, such as Hassan [24]. This literature focuses on risk-sharing effects related to country size as reflected in GDP, whereas we focus on the coordination effects driven by the size of a country's debt.

[2] Market orders avoid the thorny theoretical issue of investors using the information aggregated by the market clearing price to decide which country to invest in, a topic extensively studied in the literature on Rational Expectations Equilibrium.

available to investors, all savings of investors go to buy these sovereign bonds. This implies via the market clearing condition that

$$s_1 p_1 + s_2 p_2 = p_1 + s p_2 = 1 + f.$$

Country i has fundamentals denoted θ_i. Purely as a matter of notation we write the fiscal surplus as proportional to size and fundamentals, i.e., for country i it is $s_i\theta_i$. Then, country i has resources available for repayment consisting of the fiscal surplus $s_i\theta_i$ and the proceeds from newly issued bonds $s_i p_i$, for a total of $(s_i\theta_i + s_i p_i)$. We assume that a country defaults if and only if[3]

$$\underbrace{s_i\theta_i + s_i p_i}_{\text{total funds available}} \quad < \quad \underbrace{s_i}_{\text{debt obligations}} \quad \iff p_i < (1 - \theta_i). \tag{1}$$

If a country defaults at date 0, there is zero recovery and any investors who purchased the bonds of that country receive nothing.[4] If a country does not default, then each bond of that country pays off one at date 1. For simplicity, there is no default possibility at date 1, e.g., this assumption can be justified by a sufficiently high fundamental in period 1.

We note that our model of sovereign debt features a multiple equilibrium crisis, in the sense of Calvo [9] and Cole and Kehoe [12]. If investors conjecture that other investors will not invest in the debt of a given country, then p_i is low which means the country is more likely to default, which rationalizes the conjecture that other investors will not invest in the debt of the country.

The "fundamentals" of θ_i increase a country's surplus thus giving the country more cushion against default. For most of our analysis we refer to θ_i as the country's fiscal surplus, which then increases the funds available to the country to roll over its debt. But there are other interpretations which are in keeping with our modeling. For the case of foreign currency denominated debt, θ_i can include both the fiscal surplus and the foreign reserves of the country. For the case where the debt is denominated in domestic currency, θ_i can include resources the central bank may be willing

[3]One can think of the timing, as discussed in the text, as s_i is past debts that must be rolled over. This is a rollover risk interpretation, where we take the past debt as given. Here is another interpretation. The bonds are auctioned at date 0 with investors anticipating repayment at date 1. The date 0 proceeds of $s_i p_i$ are used by the country in a manner that will generate $s_i\theta_i + s_i p_i$ at date 1 which is then used to repay the auctioned debt of s_i. He et al. [29] discuss the difference between old debt and new debt in more detail.

[4]We study the case of positive recovery in Section 3.6.

to provide to forestall a rollover crisis. In this case, such resources, provided via monetization of debt, may be limited by central bank concerns over inflation or a devalued exchange rate (and its potential negative effects on the country's real surplus). Finally, θ_i can also be interpreted to include reputational costs associated with defaulting on debts, in which case the default equation, $s_i p_i < s_i(1-\theta_i)$, can be read as one where default is driven by unwillingness-to-pay.

We follow the global games approach to link equilibrium selection to fundamentals. We assume that there is a publicly observable world-level fundamental index $\theta \in (0,1)$. Our analysis focuses on a measure of relative strength between country 1 and country 2, which we denote by $\tilde{\delta}$ and is publicly unobservable. Specifically, conditional on the relative strength $\tilde{\delta}$, the fundamentals of these two countries satisfy

$$1-\theta_1\left(\tilde{\delta}\right) = (1-\theta)\exp\left(-\tilde{\delta}\right) \qquad \text{and} \qquad 1-\theta_2\left(\tilde{\delta}\right) = (1-\theta)\exp\left(\tilde{\delta}\right). \tag{2}$$

Recall from (1) that $1-\theta_i$ is the funding need of a country. Given $\tilde{\delta}$, the higher the θ, the greater the surplus of both countries and therefore the lower their funding need. And, given θ, the higher the $\tilde{\delta}$, the better are country 1 fundamentals relative to country 2, and therefore the lower is country 1's relative funding need.[5] Finally, the above specification implies that the funding need for each country is always positive.

We assume that the relative strength of country 1, has a support $\tilde{\delta} \in \left[-\overline{\delta}, \overline{\delta}\right]$. We do not need to take a stand on the distribution over the interval $\left[-\overline{\delta}, \overline{\delta}\right]$. Unless specified otherwise, we assume $\overline{\delta} < \ln \frac{1+f}{s(1-\theta)}$, which ensures that for the worst case scenario, financing need of the weaker country exceeds the total savings $1+f$. This gives us the usual dominance regions when the fundamentals take extreme values.

As we will use the global games technique to pin down the unique threshold strategy equilibrium, we assume that country 1's relative strength $\tilde{\delta}$ is not publicly observable. Instead, each investor $j \in [0,1]$ receives a private signal

$$\delta_j = \tilde{\delta} + \epsilon_j,$$

where $\epsilon_j \sim \mathbb{U}[-\sigma, \sigma]$ and ϵ_j are independent across all investors $j \in [0,1]$. Following the global

[5] The scale of $1-\theta$ and exponential noise $e^{\tilde{\delta}}$ and $e^{-\tilde{\delta}}$ in (2) help in obtaining a simple closed-form solution in our model, as shown shortly. The Appendix B.1 considers an additive specification $\theta_i = \theta + (-1)^i \tilde{\delta}$ and solves the case for $\sigma > 0$; we show that the main qualitative results hold in that setting.

games literature a la Morris and Shin [38] we will focus on the limit case where the noise vanishes, i.e., $\sigma \to 0$.

Finally, note that although we do not need to take a stand on the distribution of $\tilde{\delta}$, for much of the analysis, it will make most sense to think of a distribution that places all of the mass around some point δ_0 and almost no mass on other points. This will correspond to a case where investor-j is almost sure that fundamentals are δ_0, but is unsure about what other investors know, and whether other investors know that investor-j knows fundamentals are δ_0, and so on. In other words, in the limiting case fundamental uncertainty vanishes and only strategic uncertainty remains.

2.2 Equilibrium Characterization and Properties

We focus on symmetric threshold equilibria in this section. More specifically, we assume that all investors adopt the same threshold strategy in which each investor purchases country 1 bonds if and only if his private signal about country 1's relative strength is above a certain threshold, i.e. $\delta_j > \delta^*$; otherwise the investor purchases country 2 bonds. We will later show in Proposition 2 that if we restrict agents to monotone strategies, i.e. strategies in which an agent's investment in a country is weakly increasing in the signal received about that country, the symmetric threshold equilibrium is the unique equilibrium. Later in this paper, we study non-monotone strategies, which can exist for some parameters, and describe a novel class of equilibria.

Deriving the equilibrium threshold. In equilibrium, the marginal investor who receives the threshold signal $\delta_j = \delta^*$ must be indifferent between investing his money in either country. Based on this signal, the marginal investor forms belief about other investors' signals and hence their strategies. Denote by x the fraction of investors who receive signals that are above his own signal $\delta_j = \delta^*$, and as implied by threshold strategies will invest in country 1. It is well-known (e.g., Morris and Shin [38]) that in the limit of diminishing noise $\sigma \to 0$, the marginal investor forms a "diffuse" view about other investors' strategies, in that he assigns a uniform distribution for $x \sim \mathbb{U}[0,1]$.

Combined with the threshold strategy, the fraction of investors who purchase the bonds of country 1 is equal to the fraction of investors deemed more optimistic than the marginal agent, x. Thus, the total funds going to country 1 and 2 are $(1+f)\,x$ and $(1+f)\,(1-x)$, respectively. The

resulting bond prices are thus

$$p_1 = (1+f)\,x \qquad \text{and} \qquad p_2 = \frac{(1+f)\,(1-x)}{s}.$$

We now calculate the expected return from investing in bond i, Π_i.

Expected return from investing in country 1. Given x and its fundamental θ_1, country 1 does not default if and only if

$$p_1 - 1 + \theta_1 \;=\; (1+f)\,x - 1 + \theta_1 \geq 0 \iff x \geq \frac{1-\theta_1}{1+f}. \tag{3}$$

This is intuitive: country 1 does not default only when there are sufficient investors who receive favorable signals about country 1 and place their funds in country 1's bonds accordingly. The survival threshold $\frac{1-\theta_1}{1+f}$ is lower when country 1's fundamental, θ_1, is higher and when the total funds available for savings, f, are higher.

Of course, country 1's fundamental $1-\theta_1 = (1-\theta)\,e^{-\tilde{\delta}}$ in (2) is uncertain. We take the limit as $\sigma \to 0$, so that the signal is almost perfect and the threshold investor who receives a signal δ^* will be almost certain that[6]

$$1-\theta_1 = (1-\theta)\,e^{-\delta^*}. \tag{4}$$

Hence, in the limiting case of $\sigma \to 0$, plugging (4) into (3) we find that the large country 1 survives if and only if

$$x \geq \frac{1-\theta_1}{1+f} = \frac{(1-\theta)\,e^{-\delta^*}}{1+f}. \tag{5}$$

Here, either higher average fundamentals θ or a higher threshold δ^* make country 1 more likely to repay its debts.

Now we calculate the investors' return by investing in country 1. Conditional on survival, the

[6] In equilibrium, θ_1 depends on the realization of x, which is the fraction of investors with signals above δ^*. Given that the signal noise ϵ_j is drawn from a uniform distribution over $[-\sigma, \sigma]$, we have

$$x = \Pr\left(\tilde{\delta} + \epsilon_j > \delta^*\right) = \frac{\tilde{\delta} + \sigma - \delta^*}{2\sigma} \Rightarrow \tilde{\delta} = \delta^* + (2x-1)\,\sigma.$$

which implies that $\theta_1 = \theta + (1-\theta)\left(1 - e^{-\delta^* - (2x-1)\sigma}\right)$. Taking $\sigma \to 0$ we get (4).

realized return is

$$\frac{1}{p_1} = \frac{1}{(1+f)\,x},$$

while if default occurs the realized return is 0. From the point of view of the threshold investor with signal δ^*, the chance that country 1 survives is simply the integral with respect to the uniform density dx from $\frac{(1-\theta)e^{-\delta^*}}{1+f}$ to 1:

$$\Pi_1(\delta^*) = \int_{\frac{(1-\theta)e^{-\delta^*}}{1+f}}^{1} \frac{1}{(1+f)\,x} dx = \frac{1}{1+f}\left(\ln\frac{1+f}{1-\theta} + \delta^*\right). \tag{6}$$

The higher the threshold δ^*, the greater the chance that country 1 survives, and hence the higher the return by investing in country 1 bonds.

Expected return from investing in country 2. Denote the measure of investors that are investing in country 2 by $x' \equiv 1 - x$, that is the fraction of investors that are more pessimistic than the marginal agent, which again follows a uniform distribution over $[0, 1]$. If the investor instead purchases country 2's bonds, he knows that country 2 does not default if and only if

$$sp_2 - s + s\theta_2 = (1+f)\,x' - s + s\theta_2 \geq 0 \Leftrightarrow x' \geq \frac{s\,(1-\theta_2)}{1+f}, \tag{7}$$

Country 2 survives if the fraction of investors investing in country 2, x', is sufficiently high. The threshold is lower if the country is smaller, fundamentals are better, and the total funds available for savings are higher.

Similar to the argument in the previous section, in the limiting case of almost perfect signal $\sigma \to 0$, country 2 fundamental θ_2 in (7) is almost certain from the perspective of the threshold investor with signal δ^* (recall (2)):

$$1 - \theta_2 = (1-\theta)\,e^{\delta^*}. \tag{8}$$

Plugging equation (8) into equation (7), we find that country 2 survives if and only if

$$x' \geq \frac{s\,(1-\theta)\,e^{\delta^*}}{1+f}. \tag{9}$$

Relative to (5), country size s plays a role. All else equal, the lower size s and the smaller country

2, the more likely that the country 2 survives.

Given survival, the investors' return of investing in country 2, conditional on x', is

$$\frac{1}{p_2} = \frac{s}{(1+f)\,x'}; \tag{10}$$

while the return is zero if country 2 defaults. As a result, using (10), the expected return from investing in country 2 is

$$\Pi_2(\delta^*) = \int_{\frac{s(1-\theta)e^{\delta^*}}{1+f}}^{1} \frac{s}{(1+f)\,x'} dx' = \frac{1}{1+f} \cdot s \left(-\ln s + \ln \frac{1+f}{1-\theta} - \delta^* \right) \tag{11}$$

Note that if $s = 1$, we see that this profit is the same as for country 1 whose debt size is fixed at 1.

Expected return of investing in country 1 versus country 2. Figure 1 plots the return to investing in each country as a function of x (x') which is the measure of investors investing in country 1 (country 2). For illustration, we take the hypothetical equilibrium threshold $\delta^* = 0$, and study the payoffs from the perspective of the marginal investor with $\tilde{\delta} = \delta^* = 0$ so $\theta_1 = \theta_2 = \theta$. Consider the solid green curve first which is the return to investing in country 1. For x below the default threshold $\frac{1-\theta}{1+f}$, the return is zero. This default threshold is relatively high, since country 1 is large and hence it needs a large number of investors to buy bonds to ensure a successful auction. Across the threshold $\frac{1-\theta}{1+f}$, investor actions are strategic complements – i.e., if a given investor knows that other investors are going to invest in country 1, the investor wants to follow suit. Past the threshold, the return falls as the face value of bonds is constant and investors' demand simply bids up the price of the bonds. In this region, investor actions are strategic substitutes. The marginal investor's expected return from investing in country 1 is the integral of shaded area beneath the green solid line.

The dashed red curve plots the return to investing in country 2, as a function of x' which is the measure of investors investing in country 2. The default threshold for country 2, which is $\frac{s(1-\theta)}{1+f}$, is lower than for country 1 ($\frac{1-\theta}{1+f}$) because country 2 only needs to repay a smaller number of bonds. When $\delta^* = 0$, i.e., the marginal investor with signal $\delta^* = 0$ believes that both countries share the same fundamentals, the threshold return to investing in country 2 is $\frac{1}{1-\theta}$. This is the same as the threshold return to investing in country 1, as shown in Figure 1. While country 2 has a lower

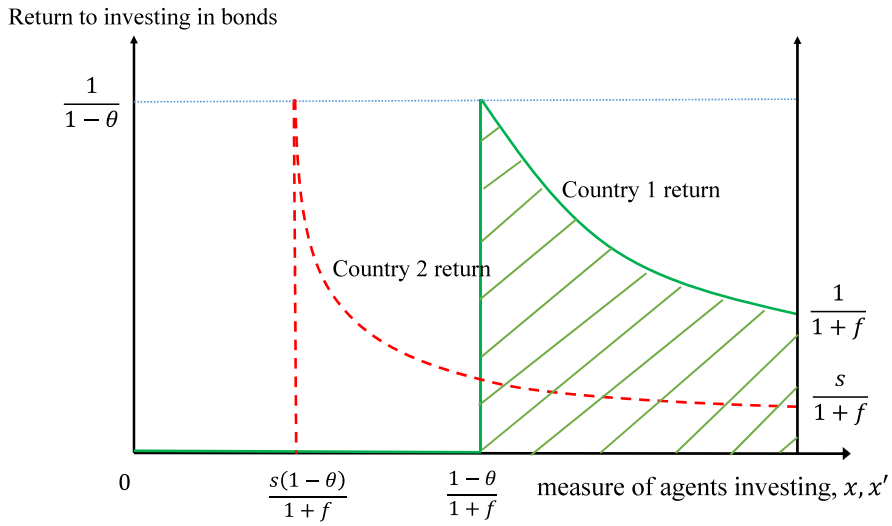

Figure 1: **Returns of the marginal investor when investing in country 1 (country 2) as a function of x (x').** The return to investing in country 1 (2) is the green solid (red dashed) line. We assume $\delta^* = 0$ so that the marginal investor with $\tilde{\delta} = \delta^* = 0$ believes that both countries have the same fundamentals. The bonds issued by the large country 1 (small country 2) only pay when $x > \frac{1-\theta}{1+f}$ ($x' > \frac{s(1-\theta)}{1+f}$). The return to country 1's bonds falls to $\frac{1}{1+f}$ when $x = 1$, while for country 2's bonds the return falls more rapidly to $\frac{s}{1+f}$ when $x' = 1$.

default threshold which implies a smaller strategic complementarity effect, past the threshold the return to investing in country 2 falls off quickly. That is, the strategic substitutes effect is more significant for country 2 than country 1. This is because country 2 has a small bond issue and hence an increase in demand for country 2 bonds increases the bond price (decreases return) more than the same increase in demand for country 1 bonds. We see this most clearly at the boundary where $x = x' = 1$, where the return to investing in the large country 1 is $\frac{1}{1+f}$, while the return to investing in country 2 is $\frac{s}{1+f}$.

To sum up, because the large country auctions off more bonds, it needs more investors to participate to ensure no-default. However, the very fact that the large country sells more bonds makes the large country a deeper financial market that can offer a higher return on investment. This tradeoff – size features more rollover risk but provides a more liquid savings vehicle – is at the heart of our analysis.

Equilibrium threshold δ^*. The equilibrium threshold δ^* is determined by the indifference condition for the threshold investor between investing in these two countries. Setting $\Pi_1(\delta^*) - \Pi_2(\delta^*) = 0$,

plugging in (6) and (11), the equilibrium threshold signal δ^* is given by

$$\delta^*(s,z) = \underbrace{-\frac{1-s}{1+s}}_{\text{liquidity, }(-)} \cdot z + \underbrace{\frac{-s\ln s}{1+s}}_{\text{rollover risk, }(+)} \qquad \text{where} \qquad z \equiv \ln\frac{1+f}{1-\theta} > 0. \tag{12}$$

Here, z measures aggregate funding conditions, which is greater if either more aggregate funds f are available or there is a higher aggregate fundamental θ. The "savings glut" which many have argued to characterize the world economy for the last decade is a case of high z.

From (12) we see that there are two effects of size. The first term is negative (for $s \in (0,1)$) and reflects the liquidity or market depth benefit that accrues to the larger country, making country 1 safer all else equal. The second term is positive and reflects the rollover risk for country 1, whereby a larger size makes country 1 less safe. The benefit term is modulated by the aggregate funding condition z. We next discuss implications of our model based on the equation (12).

3 Model Implications

3.1 Determination of asset safety

Comparing the realized fundamental $\tilde{\delta}$ to the equilibrium threshold δ^* tells us which of the two countries will not default, and thus which country's debt will serve as the safe store of value. Consider the case where the distribution of $\tilde{\delta}$ places all of the mass around some point δ_0 and almost no mass on other points. This corresponds to a case where investor-j is almost sure that fundamentals are δ_0, but is unsure about what other investors know, and whether other investors know that investor-j knows fundamentals are δ_0. If $\delta_0 > \delta^*$ then country 1 debt is the safe asset, while if $\delta_0 < \delta^*$ then country 2 debt is the safe asset. Given that all investors know almost surely the value of δ_0, investors are then almost sure which debt is safe. Mapping this interpretation to thinking about the world, the model says today may be a day that US Treasury bonds are almost surely safe, i.e., $\delta_0 >> \delta^*$. But there may be a news story out that questions the fundamentals of the US (e.g., negotiations regarding the debt limit), and while investor-j may know that it is still the case that $\delta_0 >> \delta^*$, the failure of common knowledge establishes a lower bound δ^* at which the US Treasury bond will cease to be safe.

The following proposition gives the properties of the equilibrium threshold $\delta^*(s,z)$, as a function of country 2's relative size s and the aggregate funding condition z.

Proposition 1 *We have the following results for the equilibrium threshold $\delta^*(s,z)$.*

1. *The equilibrium threshold $\delta^*(s,z)$ is decreasing in the aggregate funding conditions z. Hence, country 1's bonds can be the safe asset for worse values of country 1 fundamentals $\tilde{\delta}$, if the aggregate fundamental θ or aggregate saving f is higher.*

2. *The equilibrium threshold $\delta^*(s,z) \leq 0$ for all $s \in (0,1]$, if and only if $z \geq 1$. Hence, when the aggregate funding $z \geq 1$, the bonds issued by the larger country 1 can be the safe asset for worse values of country 1 fundamentals $\tilde{\delta}$.*

3. *When $s \to 0$ the equilibrium threshold $\delta^*(s,z)$ approaches its minimum, i.e., $\lim_{s\to 0}\delta^*(s,z) = \inf_{s\in(0,1]}\delta^*(s,z) = -z < 0$. This implies that all else equal, country 1 is the safe asset over the widest range of fundamentals when country 2 is smallest.*

Proof. Result (1.) follows because of $\frac{\partial}{\partial z}\delta^*(s,z) = -\frac{1-s}{1+s} < 0$. To show result (2.), note that when $z=1$ we have $\delta^*(s,z=1) = \frac{s-s\ln s-1}{1+s} < 0$ for $s \in (0,1]$. This inequality can be shown by observing (i) $[s - s\ln s - 1]' > 0$ and (ii) $[s - s\ln s - 1]_{s=1} = 0$. Result (3.) holds because

$$\delta^*(s,z) = -\frac{1-s}{1+s}z + \frac{-s\ln s}{1+s} > -\frac{1-s}{1+s}z > -z,$$

where the last inequality is due to $-\frac{1-s}{1+s}z$ being increasing in s for $z > 0$. ■

We illustrate these effects in Figure 2. The left panel of Figure 2 plots δ^* as a function of s for the case of $z=1$, which corresponds to strong aggregate funding conditions with abundant savings and/or good fundamentals. In this case, the equilibrium threshold $\delta^*(s)$ is always negative, and is monotonically increasing in the small country size s. For small s close to zero, the large country is safe even for low possible values of the fundamental $\tilde{\delta}$, because in this case country 2 does not exist as an investment alternative. Then because all investors have no choice but to invest in country 1, the bonds issued by country 1 have minimal rollover risk. If we assume that the aggregate savings $1+f$ are enough to cover country 1's financing shortfall $1-\theta_1\left(\tilde{\delta}\right)$ even for the worst realization of $\tilde{\delta} = -\overline{\delta}$ then country 1 will always be safe in this case. This $s=0$ case offers one perspective

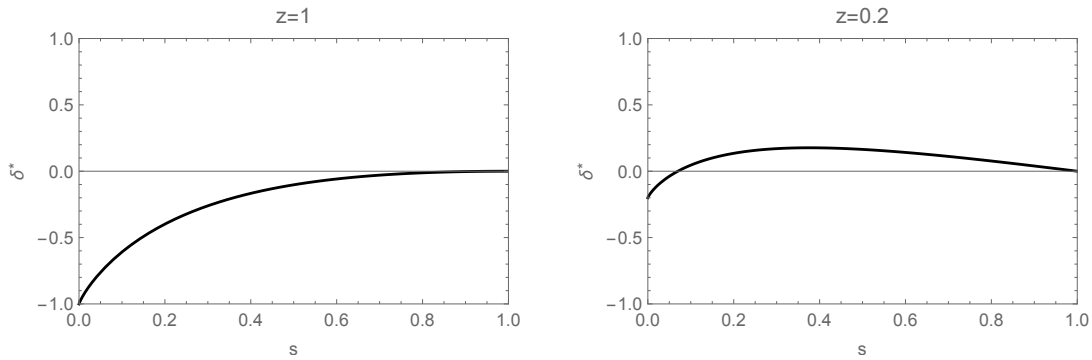

Figure 2: **Equilibrium threshold δ^* as a function of country 2 size s.** The left panel is for the case of strong aggregate funding conditions with $z = 1$, and the right panel is for the case of weak aggregate funding conditions with $z = 0.2$.

on why Japan has been able to sustain a large debt without suffering a rollover crisis. Many of the investors in Japan are so heavily invested in Japanese government, eschewing foreign alternative investments, making Japan's debt safe. In the model, when $s = 0$, investors have no elsewhere to go and are forced into a home bias. If this home bias in investment disappeared, then Japanese debt may no longer be safe.

The right panel in Figure 2 plots δ^* for a case of weak aggregate funding conditions ($z = 0.2$), with insufficient savings and/or low fundamentals. Consistent with the first result in Proposition 1 we see that in this case the large country can be at a disadvantage. For medium levels of s (around 0.4), investors are concerned that there will not be enough demand for the large country bonds, exposing the large country to rollover risk. As a result, investors coordinate investment into the small country's debt. Note that this may be the case even if the small country has worse fundamentals. For small s, the size disadvantage of the small country becomes a concern, and the large country is safe even with poor fundamentals (the third result in Proposition 1). For s large, we are back in the symmetric case. Comparing the right panel with $z = 1$ to the left panel with $z = 0.2$ highlights that the large country's debt size is an *unambiguous* advantage *only* when the aggregate funding conditions are strong; as the pool of savings shrink, the large debt size triggers rollover risk fears so that investors coordinate investment into the small country's debt.

3.2 Relative fundamentals

Our model emphasizes relative fundamentals as a central ingredient in debt valuation. To clarify this point, consider a standard model without coordination elements and without the safe asset saving need. In particular, suppose that the world interest rate is R^* and consider any two countries in the world with surpluses given by θ_1 and θ_2. Suppose that investors purchase these countries' bonds for $p_i s_i$ and receive repayment of $s_i \min(\theta_i, 1)$. Then,

$$p_1 = \frac{\mathbb{E}\left[\min\left(\theta_1, 1\right)\right]}{1 + R^*} \quad \text{and} \quad p_2 = \frac{\mathbb{E}\left[\min\left(\theta_2, 1\right)\right]}{1 + R^*},$$

so that bond prices depend on fundamentals, but not particularly on relative fundamentals $\theta_1 - \theta_2$. In contrast, in our model if country-i has the better fundamentals (relative to the equilibrium threshold δ^*), it attracts all the savings so that

$$p_i = 1 + f \quad \text{and} \quad p_{-i} = 0. \tag{13}$$

Valuation in our model becomes sensitive to relative fundamentals, as investors endogenously coordinate to buy bonds that they deem safer. In Section 3.6 we show that these forces also explain why a safe asset carries a negative β.

The importance of relative fundamentals helps us understand why, despite deteriorating US fiscal conditions, US Treasury bond prices have continued to be high: In short, all countries' fiscal conditions have deteriorated along with the US, so that US debt has maintained and perhaps strengthened its safe asset status. The same logic can be used to understand the value of the German Bund (as a safe asset within the Euro-area) despite deteriorating German fiscal conditions. The Bund has retained/enhanced its value because of the deteriorating general European fiscal conditions.

3.3 Size and aggregate funding conditions

Our model highlights the importance of debt size in determining safety, and its interactions with the aggregate funding conditions. In the high aggregate funding regime, which the literature on the global savings glut has argued to be true of the world in recent history (see, e.g., Bernanke [4],

Caballero et al. [8], and Caballero and Krishnamurthy [6]), higher debt size increases safety. US Treasury bonds are the world safe asset in part because US has maintained large debt issues that can accommodate the world's safe asset demands.

These predictions of the model also offer some insight into when US Treasury bonds may cease being a safe asset. If the world continues in the high savings regime, the US will only be displaced if another country can offer a large debt size and/or good relative fundamentals. This seems unlikely in the foreseeable future. On the other hand, if the world switches to the low savings regime, it is possible that US Treasury bonds become unsafe, and another country debt with a smaller debt size and good fundamentals, such as the German Bund, becomes the dominant safe asset.

The size effect also offers a perspective on the period prior to World War I when the UK consol bond was the world's safe asset. For example, foreign exchange reserves around the world were largely held in consol bonds (see Eichengreen [16, 17, 18]).

Despite the fact that the GDP of the US had caught up to the GDP of the UK by 1870, the UK consol bond was the premier safe asset. This seems even more puzzling, as in 1890, the US had a lower Debt/GDP ratio than the UK (0.10 for US versus 0.43 for UK). Our model provides one explanation for this puzzle. In 1890, the absolute amount of UK Debt was about 4.3 times the size of US Debt, and the higher float of UK debt was perhaps one reason that the UK attracted safe asset demand during a period when its fundamentals were likely worse. Debt stocks of both US and UK rise quickly in World War I, with the UK Debt/GDP reaching 1.40 by 1920 (from 0.43 in 1870), and the US debt/UK debt reaching 0.46 by 1920 (from 0.23 in 1870). Our model suggests that as the UK debt size grew, size turned from a liquidity advantage to a rollover risk concern. At the same time, the rise in the US debt as a liquid and sizable alternative led investors to prefer US debt over UK debt as a safe asset.

The size effect of our model also identifies a novel *contagion* channel. In the high savings regime, increasing the debt size of the large debt country reduces δ^* and thus decreases the safety threshold of the smaller country. We can see this from Figure 2, left panel with $z = 1$. Suppose that we decrease the *relative* size of country 2, s, away from 1; it is equivalent to increasing the size of the large country's debt. We see that δ^* falls in this case. Linking this observation to data, from 2007Q4 to 2009Q4, the supply of US Treasury bonds increased by \$2.7 trillion (the money stock increased another \$1.3 trillion). Our model suggests that this increase should have hurt the safety

of other country's debts. That is, our model suggests a causal link from the increase in US Treasury bond supply/Fed QE and the eruption of the European sovereign debt crisis in 2010. Intuitively, the expansion of US debt supply created safe "parking spots" for funds that may otherwise have been invested in European sovereign debt. We develop this point further in He et al. [29].

3.4 Switzerland, Denmark, and gold

So far, there are no savings vehicles in the model other than the countries' sovereign debts. That is, all savings needs must be satisfied by sovereign debt that is subject to rollover risk. There is no "gold" in the model, nor are other companies, banks or other governments that are able to honor commitments to repay debts. In practice such assets do exist. Switzerland and Denmark have been prominent in the news in 2015 because of safe-haven flows into these countries, perhaps because these countries can commit to repay their relatively small outstanding supply of bonds. It is easy to accommodate this possibility into the model.

Suppose that there exists a quantity of full-commitment sovereign bonds. The supply of these bonds is $\underline{s}$, that is, these bonds pay in total $\underline{s}$ at the final date. Investors invest $f - \hat{f}$ in these bonds, with a return of $\frac{\underline{s}}{f-\hat{f}}$. Let us focus on the symmetric case with $s = 1$ and thus $\delta^* = 0$. Investing in sovereign bonds of country 1 or country 2 depending on the signal $\tilde{\delta}$ gives a return of $\frac{1}{1+\hat{f}}$ as the small noise assumption implies that investors are perfectly able to pick the "winner". Thus in equilibrium it must be that the full-commitment bond also offers a return of $\frac{1}{1+\hat{f}}$, which then implies that

$$\frac{\underline{s}}{f-\hat{f}} = \frac{1}{1+\hat{f}} \Rightarrow \hat{f} = \frac{f-\underline{s}}{1+\underline{s}}.$$

Assume that the supply of full-commitment bonds $\underline{s}$ satisfies $\underline{s} < f$ so that $\hat{f} > 0$. We then can solve our model following exactly the same steps, only with f redefined as $\hat{f}$. Thus, the model can be interpreted as one where alternative savings vehicles do exist, but their supplies are such that substantially most of the world's safe asset needs must still be satisfied by debt that is subject to rollover risk.[7]

[7] The total government debt of Switzerland in early 2015 was \$127bn. Its central bank liabilities were near \$500bn, having grown significantly with the Europe crisis and the Swiss decision to maintain their exchange rate vis-a-vis the Euro. Total government debt in Denmark was \$155bn. Total central bank holdings of gold around the world are approximately \$1.2tn, although this amount is largely backing for government liabilities, rather than privately investable gold. It is difficult to get a clear sense of the quantity of gold held privately as an investment, but it is

Denmark and Switzerland have recently restricted their supplies of safe bonds. The result has been that the prices of their bonds have risen, with interest rates in both countries falling below zero. We can also see this in our model. Reducing $\underline{s}$ causes $\hat{f}$ to rise, and hence the price of safe bonds rises.

3.5 Non-monotone strategies and joint safety equilibria

So far we have restricted the agents' strategy space to so-called "threshold" strategies, i.e., invest in country 1 if δ_j is above certain threshold; otherwise invest in country 2. This section discusses potential equilibria once this strategy space is relaxed.

Denote the probability (or fraction) of investment in country 1 by an agent with signal δ_j by $\phi(\delta_j) \in [0,1]$; the agent's strategy is monotone if $\phi(\delta_j)$ is monotonically increasing in his signal δ_j of the country 1's fundamental, i.e., $\phi'(\delta_j) \geq 0$. Then we have the following Proposition proved in Appendix B.2:

Proposition 2 *The equilibrium with threshold strategies constructed in Eq. (12) is the unique equilibrium within the monotone strategy space.*

If we allow agents to choose among non-monotone strategies, i.e. $\phi(\delta_j)$ is non-monotone, then for large enough z it is possible to construct equilibria where both countries are safe for some values of the relative fundamental signal $\tilde{\delta}$ (while one country fails if $\tilde{\delta}$ is too low or too high). In Appendix A.1 we construct a non-monotone equilibrium in which agents use "oscillating strategies" that are a tractable way of approximating mixed strategies.

Under this oscillating strategy, agents invest in country 2 for sufficiently low δ_j. If the signal is slightly above an endogenous threshold δ_L, agents then invest in country 1, but go back to investing in country 2 for higher signals, oscillating back and forth. Oscillation stops when signals reach another endogenous threshold δ_H, above which agents always invest in country 1. The oscillation intervals are increasing functions of σ, so that when $\sigma \to 0$ this strategy approximates mixed strategies. Such a strategy is driven by the strategic substitution effect in our model, as it serves to equalize returns from investing when both countries are safe. Indeed, in the constructed equilibria

likely not larger than the central bank holdings of \$1.2tn. The most liquid gold investment are gold ETFs. Total capitalization of US gold ETFs was \$39bn in early 2015. As a comparison, the total supply of Treasury bonds plus central bank liabilities (reserves, cash, repos) in early 2015 was over \$16tn.

with oscillating strategies, non-monotonicity occurs only in the region where both countries are safe given the realization of fundamental $\tilde{\delta}$ and equilibrium investment strategies. In this region, knowing that both countries will be safe, investors who are indifferent oscillate between investing in country 1 and country 2 depending on their private signal realizations. That is, as the fundamental $\tilde{\delta}$ (and thus the private signal) is no longer payoff relevant for safe countries (recall that absent default the payoff of any bond is capped at one), oscillation leads to investment in exactly the proportions that equalize equilibrium returns.[8]

Though seemingly exotic, it is interesting that equilibria with such non-monotone strategies lead to the economically plausible situation that both countries' debts may be safe when z is high. This possibility cannot emerge in the case of monotone strategies in which one country always survives and one country always defaults.

All key qualitative properties in Proposition 1 derived under the threshold strategy equilibria are robust to considering the non-monotone oscillation strategy equilibria, with minor modifications. The next proposition summarizes the results parallel to Proposition 1.

Proposition 3 *We have the following results for the equilibrium with oscillating strategies.*

1. *For sufficiently favorable aggregate funding conditions $z \geq \underline{z} > 0$ where $\underline{z}$ is derived in Appendix A.1, the equilibrium with oscillating strategies exists. Oscillation (and thus joint survival) occurs on an interval*

$$[\delta_L, \delta_H] = \left[-z + (1+s)\ln(1+s) - s\ln s, z - \frac{1+s}{s}\ln(1+s)\right] \tag{14}$$

2. *The survival region of the larger country 1, $\left[\delta_L, \overline{\delta}\right]$, increases with the aggregate funding conditions z. However, a higher z also increases the survival region of the smaller country 2, $\left[-\overline{\delta}, \delta_H\right]$.*

3. *When $z \geq \underline{z}$, the bonds issued by the larger country 1 are a safe asset for a wider range of fundamentals than the bonds issued by the smaller country 2.*

[8] Every agent (say with a signal δ_i) in this region knows that other agents whose private signals span an interval of 2σ are oscillating in such a way to keep the proportions constant.

4. *All else equal, the larger country 1 is a safe asset for the lowest level of fundamentals when the debt size of country 2 goes to zero, i.e.* $s \to 0$.

The first result shows that there is a simple closed form solution for the δ_L and δ_H. Regarding the second result, recall that in the monotone threshold equilibria studied in Proposition 1, a higher z *increases* the survival region of the larger country 1 and at the same time *decreases* the survival region of the smaller country 2. This is because only one country survives in the monotone threshold equilibria. In contrast, in the oscillating strategy equilibria, both counties may survive, and thus improved aggregate funding conditions makes both countries safer. The first and third result of Proposition 3 are similar to the results of Proposition 1, i.e., under sufficiently favorable aggregate funding conditions so that the non-monotone strategy equilibrium exists, the bonds of the larger country are safer than the bonds of the smaller country. The fourth result is identical to that of Proposition 1.

3.6 Negative β safe asset

At the height of the US financial crisis, in the aftermath of the Lehman failure, the prices of US Treasury bonds increased dramatically in a flight to quality. Over a period in which the expected liabilities of the US government likely rose by several trillion dollars, the value of US government debt went up. We compute that from September 12, 2008 to the end of trading on September 15, 2008 the value of outstanding US government debt rose by just over \$70bn. Over the period from September 1, 2008 to December 31, 2008, the value of US government debt outstanding as of September 1 rose in value by around \$210bn. These observations indicate that US Treasury bonds are a "negative β" asset. In this section, we show that a safe asset in our model is naturally a negative β asset, and this β is closely tied to the strength of an asset's safety.

In our baseline model with zero recovery, the price of a safe asset is equal to $\frac{1+f}{s_i}$ regardless of shocks. This stark result does not allow us to derive predictions for the β, which is the sensitivity of price to shocks. Now we introduce a positive recovery value in default per unit of face value, $0 < l_i < 1$. This says that the total payouts from the defaulting country 1 or country 2 are l_1 or sl_2, respectively. For simplicity, we do not allow l_i to be dependent on the country's relative fundamental $\tilde{\delta}$. However, l_i may depend on the average fundamental θ, to which we will introduce

shocks later when calculating the β of the assets.

When recovery is strictly positive, there is a strong strategic substitution force that pushes investors to buy the defaulting country's debt if nobody else does so. This is because an infinitesimal investor would earn an unbounded return if she is the only investor in the defaulting country's bonds, given a strictly positive recovery. But this implies that threshold strategies are no longer optimal in any symmetric equilibrium, especially when the signal noise σ vanishes.

We thus focus on the strategy space of oscillation strategies to construct an equilibrium for the case of positive recovery. The basic idea, in the spirit of global games, is as follows. Suppose that the relative fundamental of country 1, i.e., $\tilde{\delta}$, is sufficiently high so that country 1 survives for sure, irrespective of investors' strategies. This corresponds to the upper dominance region in global games. Then, investors given their private signals will follow an oscillation strategy so that on average there are $\frac{1}{1+l_2 s}$ ($\frac{l_2 s}{1+l_2 s}$) measure of investors purchasing the bonds issued by country 1 (2). This way, the defaulting country 2 pays out $l_2 s$ while the safe country 1 pays out 1 in aggregate, and each investor receives the same return of

$$\frac{1}{(1+f)\,\frac{1}{1+l_2 s}} = \frac{l_2 s}{(1+f)\,\frac{l_2 s}{1+l_2 s}} = \frac{1+l_2 s}{1+f}.$$

For $\tilde{\delta}$'s that are below but close to the upper dominance region, we postulate that this oscillation strategy prevails in equilibrium, so that country 1 is the only safe country. On the lower dominance region (so $\tilde{\delta}$ is sufficiently low), investors follow an oscillation strategy so that on average there are $\frac{l_1}{l_1+s}$ ($\frac{s}{l_1+s}$) measures of investors purchasing the bonds issued by country 1 (2). This way, defaulting country 1 pays out l_1 while the surviving country 2 pays out s in aggregate, and each investor receives the same return of

$$\frac{l_1}{(1+f)\,\frac{l_1}{l_1+s}} = \frac{s}{(1+f)\,\frac{s}{l_1+s}} = \frac{l_1+s}{1+f}.$$

Again, $\tilde{\delta}$'s that are above but close to the lower dominance region, we postulate that this oscillation strategy prevails in equilibrium so that country 2 is the safe country.

The logic of global games suggests that there will be an endogenous switching threshold δ^*, such that it is optimal for investors with private signals above δ^* to follow the oscillation strategies in

which country 1 survives, while it is optimal for investors with private signals below δ^* to follow the oscillation strategies in which country 2 survives. When l_1, l_2 are sufficiently small, the closed-form solution for δ^* derived in Appendix B.3 is

$$\delta^* = \frac{\left[(1-l_2)\,s-(1-l_1)\right]z-(s+l_1)\ln(s+l_1)+(1+sl_2)\ln(1+l_2s)+l_1\ln l_1-sl_2\ln l_2}{(1-l_1)+s\,(1-l_2)}. \tag{15}$$

When setting $l_1 = l_2 = 0$, we recover $\delta^* = \frac{-(1-s)z-s\ln(s)}{1+s}$, our original zero-recovery monotone strategy result in (12).

For relative fundamental $\tilde{\delta} \in [\underline{\delta}, \delta^*)$, the price of each bond is given by

$$p_1 = \frac{l_1\,(1+f)}{l_1+s} \qquad \text{and} \qquad p_2 = \frac{1+f}{l_1+s}, \tag{16}$$

while for the relative fundamental $\tilde{\delta} \in (\delta^*, \underline{\delta}]$, the resulting prices are

$$p_1 = \frac{1+f}{1+l_2s} \qquad \text{and} \qquad p_2 = \frac{l_2\,(1+f)}{1+l_2s}. \tag{17}$$

Thus, this extension with a positive recovery allows us to determine the non-trivial endogenous bond prices for both countries (in the zero recovery case, those prices were zero or $(1+f)/s_i$) by equalizing the returns across both countries. As bond prices of the two countries are linked via the cash-in-the-market pricing, the defaulting country's recovery can affect the price of the safe asset.

Consider the case where $\tilde{\delta} \in (\delta^*, \underline{\delta}]$, which corresponds to the case that country 1's bonds are safe. From (17) we see that both bond prices are unaffected by l_1. In contrast, through the cash-in-the-market pricing effect, when the recovery of country 2 (l_2) decreases, p_2 drops and p_1 increases. This observation implies that the safe asset in our model will behave as a negative β asset. To see this, suppose that as aggregate fundamentals deteriorate (say θ falls), recoveries in default of both bonds, l_1 and l_2, decrease. Then, country 1's bonds gain when aggregate fundamentals deteriorate, which makes it a negative β asset, while country 2's bonds lose.

In Appendix A.2, we formally derive the β in a world with shocks to θ. Figure 3 plots the β as a function of δ. As suggested by the intuition, the β for the country 1's bonds is negative when the country 1's relative fundamental δ is high, i.e., when country 1 is the safe bond. Moreover, the higher the country 1's relative fundamental, the more negative the β of its bonds.

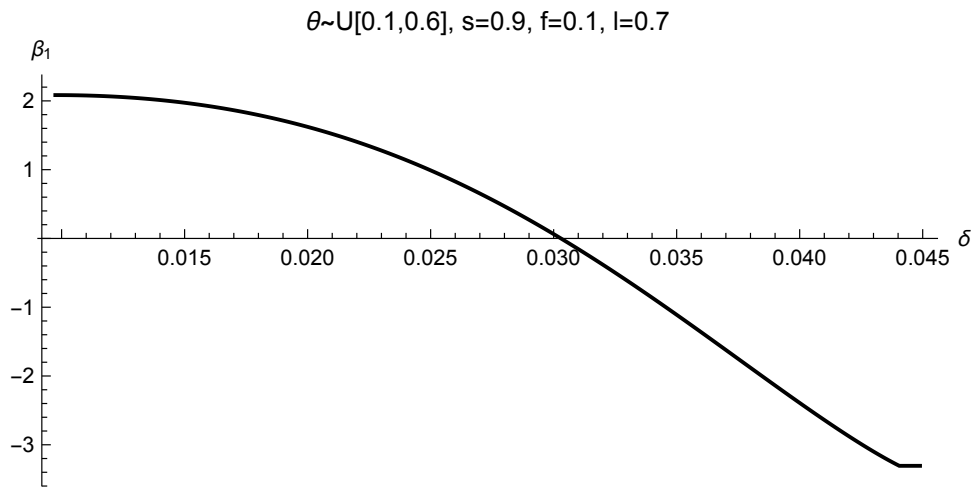

Figure 3: **Country 1 beta example:** $\beta_1 = \frac{Cov(p_1,\theta)}{Var(\theta)}$ for the bonds issued by country 1, as function of country 1's relative fundamental δ. For details, see Appendix A.2.

4 Coordination and Security Design

In this section, we characterize the benefits to coordinating through security design. We are motivated by the Eurobond proposals that have been floated over the last few years (see Claessens et al. [11], for a review of various proposals). A shared feature of these proposals is to create a common Euro-area-wide safe asset. More specifically, each country receives proceeds from the issuance of the "common bond," which is meant to serve as the safe asset. By issuing a common Euro-wide safe asset, all countries benefit from investors' flight to safety flows, as opposed to just the one country (Germany) which is the de-facto safe asset in the absence of a coordinated security design. Our model, in which the determination of asset safety is endogenous, is well-suited to analyze these issues formally. We are unaware of other similar models or formal analysis of this issue.

4.1 Main results

We assume that the two countries issue a common bond of size $\alpha\,(1+s)$ as well as individual country bonds of size $(1-\alpha)s_i$ where $s_1 = 1$ and $s_2 = s$, so that total world bond issuance in aggregate face-value is still $(1+s)$. Here, $\alpha \in [0,1]$ captures the size of common bond program.

Denote by p_c the equilibrium price for the common bonds. Since the share of proceeds from the common bond issue flowing to country i is $\frac{s_i}{1+s}$, country i receives

$$\frac{s_i}{1+s} \cdot p_c \alpha \left(1+s\right) = s_i \alpha p_c$$

from the common bond auction. Country i also issues its individual bond of size $(1-\alpha)\, s_i$ at some endogenous price p_i, so total proceeds from both common and individual bond issuances to country i are $s_i(\alpha p_c + (1-\alpha)p_i)$. Then, country i avoids default whenever,

$$s_i(\alpha p_c + (1-\alpha)p_i) + s_i\theta_i > s_i, \tag{18}$$

which is a straightforward extension of the earlier default condition (1) to include the common bond proceeds. We assume that default affects all of the country's obligations, so that a country's default leads to zero recovery on its individual bonds and its portion of common bonds. Hence, investors in common bonds receive repayments only from countries that do not default.

We model the bond auction as a two-stage game. In the first-stage, countries auction the common bonds and investors spend a total of $f - \hat{f}$ to purchase these bonds, so that the market clearing condition gives

$$f - \hat{f} = (1+s)\alpha p_c. \tag{19}$$

In this stage, $\tilde{\delta}$ is not yet observed and assumed to be distributed according to *pdf* $\left(\tilde{\delta}\right)$. In the second stage, investors use their remaining funds of $1 + \hat{f}$ to purchase individual country bonds conditional on their signal $\delta_j = \tilde{\delta} + \varepsilon_j$. After both auctions, each country makes its own default decision.

Motivated by the threshold equilibrium and oscillating equilibrium constructed in the base model, we derive the following equilibria for a setting with common bonds.

Proposition 4 *We consider two equilibria, an oscillating "maximum joint safety" equilibrium and a threshold "minimum joint safety" equilibrium. In both equilibria, the determination of asset safety depends on α as follows:*

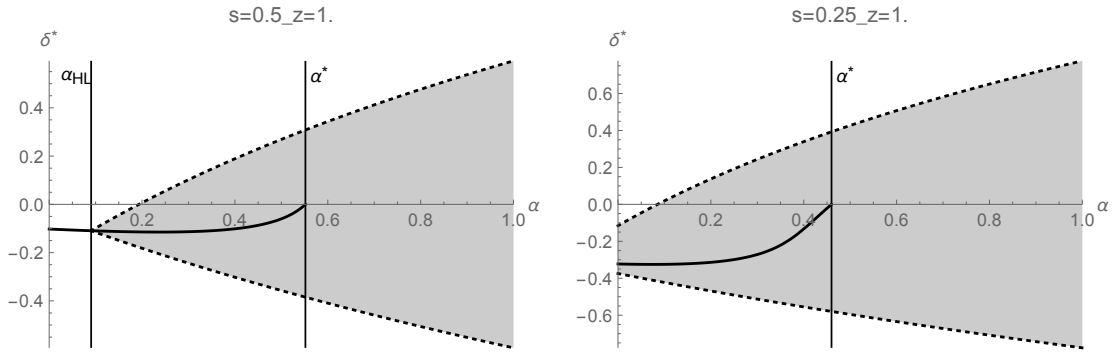

Figure 4: **Common bond equilibria:** δ^* (solid line), δ_H, δ_L (dotted lines) for the case of $s = 0.5$ and $s = 0.25$, as a function of α

1. *The threshold equilibrium exists for $\alpha \in [0, \alpha^*]$ with corresponding threshold $\delta^*(\alpha)$. If $\tilde{\delta} > \delta^*(\alpha)$, then country 1 is the safe asset and country 2 defaults, while if $\tilde{\delta} < \delta^*(\alpha)$ country 2 is the safe asset and country 1 defaults. Here, the upper bound α^* solves $\delta^*(\alpha^*) = 0$.*

2. *The oscillating equilibrium exists for $\alpha \in [\alpha_{HL}, 1]$ with corresponding lower and upper thresholds $\delta_L(\alpha)$ and $\delta_H(\alpha)$: If $\tilde{\delta} \in [\delta_L(\alpha), \delta_H(\alpha)]$, then both countries' bonds are safe, while if $\tilde{\delta} < \delta_L(\alpha)$ ($\tilde{\delta} > \delta_H(\alpha)$) country 2 is safe and country 1 defaults (country 1 is safe and country 2 default). Here, the lower bound α_{HL} solves $\delta_L(\alpha_{HL}) = \delta_H(\alpha_{HL}) = \delta^*(\alpha_{HL})$.*

Furthermore, the two thresholds satisfy $\alpha^* > \alpha_{HL}$.

Figure 4 illustrates the statement of Proposition 4 for the cases of $s = 0.5$ (left panel) and $s = 0.25$ (right panel), both for $z = 1$. The black solid line plots the threshold equilibrium cutoff δ^* for $\alpha \in [0, \alpha^*]$. As $z = 1$, we are in the high savings case illustrated in the left panel of Figure 2, and thus $\delta^*(0) < 0$. The maximum joint safety equilibrium also exists, and overlaps with the minimum joint safety equilibrium on $[\alpha_{HL}, \alpha^*]$ (with possibly negative α_{HL}). In this equilibrium, joint safety is a possibility as long as both countries do not differ too much in fundamentals. The dashed-lines in Figure 4 indicate the upper/lower bounds of the joint safety region $[\delta_L(\alpha), \delta_H(\alpha)]$, where the region itself is indicated by the grey shading.

Focusing first on the left panel with $s = 0.5$, we note that δ^* might decrease with α (one can see it graphically for small α's). This implies that the small country can actually be hurt by the introduction of small scale common bond issues. We discuss the intuition of this result later in

Section 4.2. Next, we see that the joint safety region begins at $\alpha = \alpha_{HL} > 0$, and expands as a function of α. Intuitively, as α increases, the minimum funding of the small country increases, relaxing the winner-takes-all coordination game, which in turn allows the small country to be safe for a larger range of realizations of $\tilde{\delta}$. Next, the right panel considers $s = 0.25$, thereby reducing the aggregate funding requirements for joint safety. This reduction in aggregate funding requirements is strong enough so that the maximum joint safety equilibrium exists even for $\alpha = 0$ (i.e., even in the absence of common bonds).

To sum up, our analysis in this section suggests that increases in common bond issuance, i.e., increases in α, only create Pareto gains (when gains are thought of in terms of increasing country safety) when $\alpha > \alpha^*$. In this case, increases in α raise the safety of both country 1 and country 2. For $\alpha < \alpha^*$ and in the minimum safety equilibrium, a greater α reduces safety of one country while increasing safety of the other country. Thus, small steps towards a fiscal union could be worse than no step. The rest of this section derives the equilibrium and results in Proposition 4, with proofs in Appendix A.3 and A.4.

4.2 Minimum joint safety equilibrium: threshold equilibrium

We first focus on the threshold equilibrium where only one country is safe. We will find the largest α so that this threshold equilibrium can exist, which we call α^*. We also explain why it is possible for δ^* to decrease with α in this equilibrium, i.e., why it is that common bonds may hurt the small country.

Stage 2. In the second stage, investors have $1 + \hat{f}$ funds to purchase individual bonds. Consider the marginal investor with signal δ^* who considers that a fraction x of investors have signals exceeding his. Country 1 does not default if and only if,

$$\alpha p_c + (1 - \alpha)p_1 + \theta_1 > 1.$$

Since, $f - \hat{f} = (1 + s)\alpha p_c$ by (19) and $(1 - \alpha)p_1 = x(1 + \hat{f})$, we rewrite this condition as,

$$\frac{f - \hat{f}}{1 + s} + x(1 + \hat{f}) + \theta_1 > 1 \Leftrightarrow x \geq \frac{1 - \theta_1 - \frac{1}{1+s}(f - \hat{f})}{1 + \hat{f}}.$$

We again take the limit as $\sigma \to 0$ and set $1-\theta_1 = (1-\theta)e^{-\delta^*}$. Additionally, as the return to the marginal investor in investing in country 1 is $\frac{1-\alpha}{(1+\hat{f})x}$ if the country does not default (and zero recovery in default), the expected return is (when $\hat{f} = f$ and $\alpha = 0$ one recovers the profit function in (6)):

$$\Pi_1(\delta^*) = \frac{1-\alpha}{1+\hat{f}} \ln\left(\frac{1+\hat{f}}{(1-\theta)\,e^{-\delta^*} - \frac{1}{1+s}\left(f-\hat{f}\right)}\right).$$

We repeat the same steps for the profits to investing in country 2 and find,

$$\Pi_2(\delta^*) = \frac{s\,(1-\alpha)}{1+\hat{f}} \ln\left(\frac{1+\hat{f}}{s\,(1-\theta)\,e^{\delta^*} - \frac{s}{1+s}\left(f-\hat{f}\right)}\right).$$

We solve for the threshold $\delta^*(\hat{f},\alpha)$ in the same way as before, which takes α and $\hat{f}$ as given:

$$\Pi_1(\delta^*) = \Pi_2(\delta^*) \Rightarrow \delta^*(\hat{f},\alpha). \tag{20}$$

Stage 1. Next we derive $\hat{f}$ by considering Stage 1 in which investors make their investment decisions on common bonds before $\tilde{\delta}$ realizes. Under the assumed equilibrium where only one country is safe, the return to investing in the common bond, denoted by R_{com}, is,

$$R_{com} = \frac{1}{f-\hat{f}}\left[\int_{-\bar{\delta}}^{\delta^*} \alpha s \cdot pdf\left(\tilde{\delta}\right) d\tilde{\delta} + \int_{\delta^*}^{\bar{\delta}} \alpha \cdot pdf\left(\tilde{\delta}\right) d\tilde{\delta}\right]. \tag{21}$$

At the right-hand-side of (21), the denominator in front of the brackets is the total amount of funds invested in the common bond, while the term inside the brackets is the repayment on the common bonds in the cases of repayment only by country 2 and repayment only by country 1, respectively. The returns to keeping one dollar aside and investing in individual country bonds, denoted by R_{ind}, is,

$$R_{ind} = \frac{1}{1+\hat{f}}\left[\int_{-\bar{\delta}}^{\delta^*} (1-\alpha)\, s \cdot pdf\left(\tilde{\delta}\right) d\tilde{\delta} + \int_{\delta^*}^{\bar{\delta}} (1-\alpha) \cdot pdf\left(\tilde{\delta}\right) d\tilde{\delta}\right]. \tag{22}$$

Again, the denominator in the front is the total amount of funds invested in individual bonds, while the term in parentheses is the repayment on individual bonds in the cases of repayment only

by country 2 and repayment only by country 1. Note the similarity between the terms inside the brackets in (21) and (22). The similarity arises because along the nodes of country 2 defaulting or country 1 defaulting, the payoffs, state-by-state, to common bonds and individual bonds are αs_i and $(1-\alpha)s_i$. In equilibrium, the expected return from investing in common bonds in stage one must equal to that from waiting and investing in individual bonds in stage 2:

$$R_{com} = R_{ind} \Leftrightarrow \frac{\alpha}{f-\hat{f}} = \frac{1-\alpha}{1+\hat{f}} \Leftrightarrow f - \hat{f} = \alpha\left(1+f\right). \tag{23}$$

This implies that the common bond price is given by

$$p_c = \frac{f-\hat{f}}{\alpha\left(1+s\right)} = \frac{1+f}{1+s}. \tag{24}$$

irrespective of our assumptions on the distribution of $\tilde{\delta}$, $pdf\left(\tilde{\delta}\right)$. We combine equations (20) and (23) to solve for the equilibrium threshold $\delta^*(\alpha)$ as a function of common bonds size α.

When does the minimum joint safety equilibrium exist? We next consider the bound α^* so that the minimum joint safety equilibrium exists whenever $\alpha \in [0, \alpha^*]$. We assumed in our equilibrium derivation that only one country is safe (and the other country must default). However, inspecting (18) we see that as α rises, since $p_c > 0$, it may be that even a country that receives zero proceeds from selling its individual bonds can avoid default. But this would violate the equilibrium assumption that one country defaults for sure, leading to a contradiction.

Define $\theta_{def}\left(\delta\right) \equiv \max\left[\theta_1\left(\delta\right), \theta_2\left(\delta\right)\right]$, and let us look for the strongest possible country that is still assumed to default. What is the best fundamental that we can observe in a defaulting country? Clearly, the fundamental of the defaulting country when $\tilde{\delta} = \delta^*$. Then, the strongest country that is still assumed to default is given by $\theta_{def}\left(\delta^*\right)$. This country only defaults if

$$\theta_{def}\left(\delta^*\right) + \alpha p_c < 1 \Leftrightarrow \alpha \leq \frac{1+s}{1+f}\left[1 - \theta_{def}\left(\delta^*\left(\alpha\right)\right)\right].$$

Then, define α^* as the solution to

$$\alpha^* = \frac{1+s}{1+f}\left[1 - \theta_{def}\left(\delta^*\left(\alpha^*\right)\right)\right]. \tag{25}$$

In Appendix A.3, we show that the unique threshold equilibrium $\delta^*(\alpha)$ only exists for $\alpha \in [0, \alpha^*]$ where $\alpha^* = e^{-z}(1+s)$, with $\delta^*(\alpha^*) = 0$. For any $\alpha > \alpha^*$, the threshold equilibrium ceases to exist.

The effect of introducing a small quantity of common bonds. Figure 4 shows that there are situations in which $\delta^*_\alpha(0) \equiv \left.\frac{\partial \delta^*(\alpha)}{\partial \alpha}\right|_{\alpha=0} < 0$, implying that the large country gains while the small country loses when a small fraction of common bonds are issued. Interestingly, this result is against the casual intuition that common bonds should bring safety to the small country.

This result is partly driven by the simple fact that the small country receives proportionally less common bonds proceeds. Note that common bonds decreases the default threshold, i.e., the proportion of investors required to make a country safe. Return to Figure 1, this implies that the vertical lines indicating the default threshold shift to the left for both countries, while holding the conditional returns fixed. The large country gains if, starting from $\delta^*(0)$, the new area from additional safety underneath the conditional return curve is greater than the new area for the small country. For s close to zero, almost all the common bond proceeds and thus the rollover risk reduction accrue to the large country, as the small country's vertical line almost coincides with the y-axis. As a result, introducing common bonds hurts, rather than enhances, the safety of the small country.

4.3 Maximum joint safety equilibrium: oscillating equilibrium

We now construct an oscillating equilibrium in which both countries can be safe. We will further compute the minimum value of α, denoted by α_{HL}, for which this equilibrium exists. We find $\alpha_{HL} < \alpha^*$, and the resulting overlap implies that at least two equilibria exist for some parameters, as described in Proposition 4.

As discussed in Section 3.5, the possibility that both countries may be safe rules out monotone threshold strategies. Hence in this subsection we depart from monotone threshold strategies and again consider oscillating strategies.

Stage 2. The construction of the stage 2 equilibrium is given in Appendix A.4.[9] There, for given values of $\hat{f}$, p_c and α, we derive the stage 2 equilibrium oscillating interval as

$$[\delta_L, \delta_H] = \left[-\ln\left\{ \frac{1}{1-\theta}\left[\left(1+\hat{f}\right)\frac{s^s}{(1+s)^{1+s}} + \alpha p_c\right]\right\}, \ln\left\{ \frac{1}{1-\theta}\left[\frac{1+\hat{f}}{(1+s)^{\frac{1+s}{s}}} + \alpha p_c\right]\right\}\right]. \tag{26}$$

Of course, $\hat{f}$ and p_c are equilibrium values that are determined by stage 1 investment decisions.

Stage 1. With $[\delta_L, \delta_H]$ in hand, let us determine $\hat{f}$ and $p_c = \frac{f-\hat{f}}{\alpha(1+s)}$ (there are $\alpha(1+s)$ units of common bonds, and there is $f-\hat{f}$ money invested in them). Consider an $\alpha > 0$. Then, we know that the expected returns from investing in common bonds in stage 1 and investing in the best (i.e., surviving) individual country bonds in stage 2 have to be equalized. The expected return to investing in individual bonds is given by

$$R_{ind} = \frac{1-\alpha}{1+\hat{f}}\left[\underbrace{\int_{-\overline{\delta}}^{\delta_L} s\cdot pdf\left(\tilde{\delta}\right)d\tilde{\delta}}_{\text{Country 1}} + \underbrace{\int_{\delta_L}^{\delta_H}(1+s)\,pdf\left(\tilde{\delta}\right)d\tilde{\delta}}_{\text{Joint survival}} + \underbrace{\int_{\delta_H}^{\overline{\delta}} pdf\left(\tilde{\delta}\right)d\tilde{\delta}}_{\text{Country 2}}\right], \tag{27}$$

and the expected return for common bonds is given by

$$R_{com} = \frac{\alpha}{f-\hat{f}}\left[\int_{-\overline{\delta}}^{\delta_L} s\cdot pdf\left(\tilde{\delta}\right)d\tilde{\delta} + \int_{\delta_L}^{\delta_H}(1+s)\,pdf\left(\tilde{\delta}\right)d\tilde{\delta} + \int_{\delta_H}^{\overline{\delta}} pdf\left(\tilde{\delta}\right)d\tilde{\delta}\right]. \tag{28}$$

Note the similarity between these last two expressions in (27) and (28). The similarity arises because the payoffs to common bonds and individual bonds are always αs_i and $(1-\alpha)s_i$, state-by-state. Thus, equalizing returns we have

$$R_{ind} = R_{com} \Leftrightarrow \frac{\alpha}{f-\hat{f}} = \frac{1-\alpha}{1+\hat{f}} \Leftrightarrow f-\hat{f} = \alpha(1+f). \tag{29}$$

The common bond price p_c is the same as in (24):

$$p_c = \frac{f-\hat{f}}{\alpha(1+s)} = \frac{1+f}{1+s}. \tag{30}$$

[9] It is similar to Appendix A.1,which constructs the oscillating equilibrium discussed in Proposition 3.

Plugging (29) and (30) into (26), we derive the joint safety interval

$$[\delta_L, \delta_H] = \left[-\ln\left\{ \frac{e^z}{1+s}\left[\left(\frac{s}{1+s}\right)^s (1-\alpha)+\alpha\right]\right\}, \ln\left\{ \frac{e^z}{1+s}\left[\left(\frac{1}{1+s}\right)^{\frac{1}{s}} (1-\alpha)+\alpha\right]\right\}\right] \quad (31)$$

The next proposition establishes conditions for the existence of the oscillating equilibrium.

Proposition 5 *Let $z \geq \ln(1+s)$, so that there is sufficient funding for joint safety. For any given z, define α_{HL} as the solution to $\delta_H(\alpha_{HL}) = \delta_L(\alpha_{HL})$. Then, we have $\delta^*(\alpha_{HL}) = \delta_H(\alpha_{HL}) = \delta_L(\alpha_{HL})$ and $\alpha_{HL} < \alpha^*$.*

The first result states that at α_{HL}, the thresholds $\delta^*, \delta_H, \delta_L$ all coincide. On $[\alpha_{HL}, \alpha^*]$, both equilibria exists, with the oscillating equilibrium's joint safety region uniformly increasing. At $\alpha = \alpha^*$ the threshold equilibrium ceases to exists, while the oscillating equilibrium continues to exist.

5 Debt Issuance Decision

We have thus far taken debt size and country surplus as exogenous, deriving predictions regarding the safe asset based on these variables. We now endogenize these variables, assuming that each country sees a benefit to becoming the safe asset, but faces costs in choosing debt size and surplus.

5.1 Incentives and costs of altering debt size and surplus

We suppose that the countries have natural debt sizes of s_1^* and s_2^*, which we will think of as determined by local economic and fiscal conditions. For example, countries with a higher GDP will naturally have a larger stock of debt outstanding. A country can choose to alter the size of its debt, but incurs an adjustment cost. If country 1 increases its debt size to S_1 it must increase the surplus θ proportionately, via increases in the tax base, to support the larger debt issue. We assume that increasing the debt size to S_i requires raising the surplus from θ_i to $\theta_i \frac{S_i}{s_i^*}$ and that countries face an adjustment cost $c(S_i - s_i^*)$ which is increasing and convex in $S_i - s_i^*$, with $c(0) = 0$, $c'(0) = 0$.

Separately, we suppose that countries can change their surplus without altering debt size (i.e. increasing the ratio of tax revenues to outstanding debt). In particular, a country can choose an

increment δ_i, so that the country's fundamental becomes (for country 1):

$$1-\theta_1 = (1-\theta)e^{-\tilde{\delta}-\delta_1}$$

That is, for country 1, choosing $\delta_1 > 0$ decreases its funding need $1-\theta_1$ (we can likewise define a δ_2 that improves country 2's surplus). This action costs $\kappa(\delta_i)$.

We solve for $\delta^*(S_1, S_2, \delta_1, \delta_2)$ following the same analysis as in Section 2.2. The marginal investor with signal δ^* receives the following return from investing in country 1:

$$\Pi_1 = \int_{\frac{S_1(1-\theta)e^{-\delta^*}}{1+f}}^{1} \frac{S_1}{(1+f)\,x} dx = \frac{S_1}{1+f}\left(-\ln S_1 + \ln\frac{1+f}{1-\theta} + \delta^* - \delta_1\right),$$

while the return from investing in country 2 is

$$\Pi_2 = \int_0^{\frac{1+f-S_2(1-\theta)e^{\delta^*}}{1+f}} \frac{S_2}{(1+f)\,(1-x)} dx = \frac{S_2}{1+f}\left(-\ln S_2 + \ln\frac{1+f}{1-\theta} - \delta^* + \delta_2\right).$$

Setting $\Pi_1 = \Pi_2$ and solving, we find that the equilibrium threshold is

$$\delta^*\left(S_1, S_2, \delta_1, \delta_2\right) = \frac{S_2 - S_1}{S_1 + S_2} z + \frac{S_1 \ln S_1 - S_2 \ln S_2}{S_1 + S_2} - \delta_1 \frac{S_1}{S_1 + S_2} + \delta_2 \frac{S_2}{S_1 + S_2}. \tag{32}$$

where again $z \equiv \ln \frac{1+f}{1-\theta}$ measures the strength of savings/fundamentals.

By altering S_i and δ_i a country can alter δ^*. Note that a lower δ^* helps country 1 in the sense that country 1 is the safe asset (and does not default) for a wider range of fiscal fundamentals. Likewise, a higher δ^* helps country 2.

5.2 Endogenous debt size

It is clear from equation (32) that increasing δ_i benefits country-i. Thus, as we would expect a country that increases its fiscal fundamentals improves its safe asset position. The analysis here is fairly ordinary: the country chooses δ_i to balance the safe asset benefit against the cost of increasing its surplus, $\kappa(\delta_i)$.

We instead focus our analysis on endogenous debt size, which yields more nuanced results. Let us set $\delta_i = 0$. Suppose that country 1 chooses S_1 to maximize (focusing again on monotone threshold

strategies),

$$\underbrace{-\delta^*(S_1, S_2)}_{\textbf{benefit of reserve asset status}} \quad - \quad \underbrace{c(S_1 - s_1^*)}_{\textbf{adjustment cost}} \quad . \tag{33}$$

This objective can be understood as follows. The second term is the cost of adjustment. The first term captures the benefit of adjustment, i.e., the country is able to lower the minimum level of surplus needed to be the safe asset, and hence avoid default. For any cdf for $\tilde{\delta}$ and an appropriate renormalization of the cost function, the benefit is linear in $-\delta^*(S_1, S_2)$. Likewise for country 2, the objective is

$$\delta^*(S_1, S_2) - c(S_2 - s_2^*)$$

where country 2 benefits by increasing $\delta^*(S_1, S_2)$.

Introduce the function $h(S_1, S_2; z)$, which is defined as the marginal impact of S_1 on $\delta^*(S_1, S_2)$:

$$h(S_1, S_2; z) \equiv \frac{\partial \delta^*(S_1, S_2)}{\partial S_1} = \frac{1}{(S_1 + S_2)^2}\left[S_1 + S_2(\ln S_1 + \ln S_2 + 1 - 2z)\right] \tag{34}$$

Due to symmetry, the (negative) derivative of δ^* with respect to S_2 is

$$-\frac{\partial \delta^*(S_1, S_2)}{\partial S_2} = h(S_2, S_1; z) = \frac{1}{(S_1 + S_2)^2}\left[S_2 + S_1(\ln S_1 + \ln S_2 + 1 - 2z)\right].$$

As a result, the first order conditions for endogenous debt sizes satisfy,

$$h(S_1, S_2; z) = c'(S_1 - s_1^*) \quad \text{and,} \quad h(S_2, S_1; z) = c'(S_2 - s_2^*).$$

Next we analyze the property of $h(S_1 = s_1^*, S_2 = s_2^*; z)$.

5.3 Equilibrium Characterization and "Phase Diagram"

We illustrate the solution in Figure 5. We first note that if there is no cost of changing size, then the equilibrium solves

$$h(S_1, S_2; z) = h(S_2, S_1; z) = 0.$$

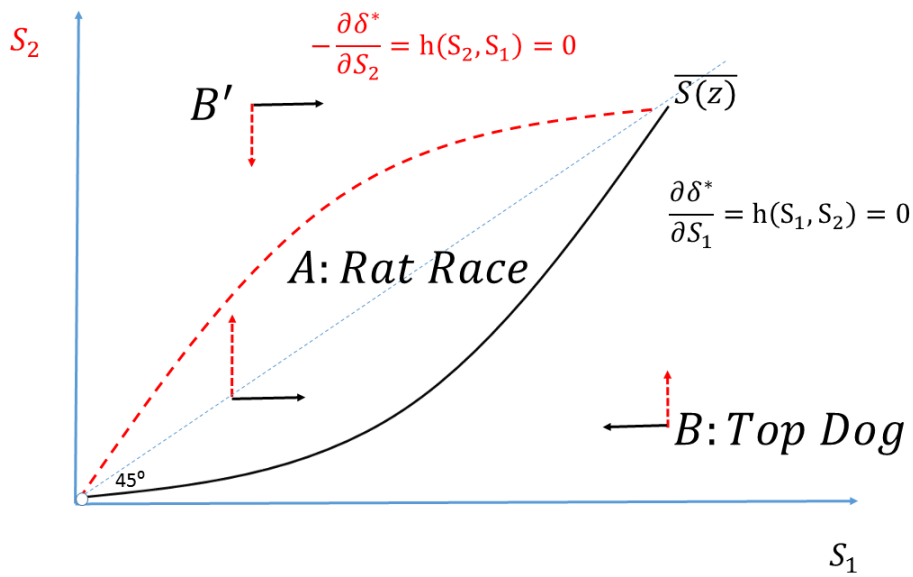

Figure 5: Diagram of incentives to alter size of debt

The equilibrium is symmetric. Define the solution in this case as $\overline{S}(z) = S_1 = S_2$, and solving we find that:

$$1 + \ln \overline{S}(z) = z \Rightarrow \overline{S}(z) = e^{z-1}.$$

On Figure 5 we plot the phase diagram where $S_1 = s_1^*$ and $S_2 = s_2^*$ are below $\overline{S}(z)$. The solid black curve corresponds to the set of points where $h(S_1, S_2; z) = 0$; that is, these are points S_1 and S_2 where changing S_1 has no effect on δ^*. The dashed red curve corresponds to the set of points where $h(S_2, S_1; z) = 0$; that is, these are points where changing S_2 has no effect on δ^*. These two curves are symmetric around the 45-degree line and cross at the point $\left(\overline{S}(z), \overline{S}(z)\right)$. The following lemma shows when country sizes are relatively small with $\max(S_1, S_2) < \overline{S}(z)$, the locus of $h(S_1, S_2; z) = 0$, which is the black curve in the figure, has $S_1 > S_2$. This implies that the black curve lies below the 45-degree line. The result is reversed for the red curve which is the locus of $-\frac{\partial \delta^*(S_1,S_2)}{\partial S_2} = h(S_2, S_1; z) = 0$ and is symmetric around the 45-degree line.

Lemma 1 *When* $\max(S_1, S_2) < \overline{S}(z)$*, the locus of* $h(S_1, S_2; z) = 0$ *has* $S_1 > S_2$*..*

Proof. We show that for $h(S_1 = S_2 + \epsilon, S_2; z) = 0$ to hold we must have $\epsilon > 0$. Using (34) we have:

$$h(S_1 = S_2 + \epsilon, S_2; z) = \epsilon + S_2(\ln(S_2 + \epsilon) - \ln S_2) + 2S_2(1 + \ln S_2 - z) = 0.$$

When $S_2 < \overline{S}(z)$, the last term $2S_2(1 + \ln S_2 - z)$ on the LHS is negative. To have $h = 0$, we must have that $\epsilon > 0$, since conditional on S_2 the sum of the first two terms is zero when $\epsilon = 0$ and increasing in ϵ. Q.E.D. ■

Intuitively, for any given level of S_2, starting from the 45-degree line country 1 has a strictly positive incentive to expand its debt size to enhance its reserve asset status, since $c'(0) = 0$. This is due to the size benefit, illustrated in the left panel of Figure 2. This explains why along the solid black curve with zero expansion incentive we must have $S_1 > S_2$. However, country 1 does not want its debts to become too large, because as illustrated by the right panel of Figure 2, becoming too large triggers rollover risk fears that may lead investors to coordinate on the smaller country's debt as the safe asset. The black curve with $h(S_1, S_2; z) = 0$ balances these two effects.

Now suppose that the initial country sizes (s_1^*, s_2^*) lies to the right of the black curve, i.e., $h(S_1 = s_1^*, S_2 = s_2^*; z) > 0$. Then in equilibrium country 1 will choose an $S_1 < s_1^*$ (exactly how much less depends on the cost function). Likewise, for any S_1 if s_2^* is above the red curve, then country 2 will choose an $S_2 < s_2^*$. For points inside the solid black and dashed red curves, both countries want to expand. The arrows on Figure 5 indicate the direction, as in a phase diagram, in which countries will change debt size given a natural size of (s_1^*, s_2^*).

There are three regions of interest in Figure 5. Region A ("RAT RACE") corresponds to a case where countries' debts are similar in size – so a roughly symmetric case. We see that in this case both countries will increase the size of their debts. This is driven by the safe asset effect: there is an externality whereby the larger country has a better safe asset position, and countries compete to become the safe asset by increasing their debt sizes. Of course, this competition to gain safe asset status is ultimately self-defeating. Take the fully symmetric case where $s_1^* = s_2^*$ and thus $\delta^* = 0$ in equilibrium. The Nash equilibrium results in countries increasing debt sizes beyond their natural sizes in a rat race, while both would be better off and save adjustment costs if they coordinate not to expand. Region B and B' ("TOP DOG") are cases with asymmetric country sizes. Take region B (as B' follows the same logic). Here country 1 is large and country 2 is relatively small.

In this case, country 1 is the top dog and not worried about losing its safe asset status, and is primarily concerned about rollover risk. In this region country 1 chooses to contract its debt size, while country 2 expands, hoping to gain some reserve status.[10]

5.4 Funding Conditions and Incentives for Expansion

By using the fact that $\frac{\partial h(\cdot,\cdot;z)}{\partial z} < 0$ in (34), we can show that the region A—which is the area inside the solid black curve and the dashed green curve—expands unambiguously as the aggregate funding condition z rises. This result has interesting implications, as it suggests that countries, once facing an environment with better aggregate funding conditions, are drawn into the rate race where they compete to offer the safe asset.

This comparative static offers a unique perspective on the expansion of relatively safe debt supplies in the period preceding the 2007 financial crisis. In the US, the government agencies, Fannie Mae and Freddie Mac, initiated a program ("Benchmark Notes") in 1998 whose purpose was to offer debt that could compete with US Treasury bonds as a large and liquid savings vehicle. Of course, the expansion of such agency debt stocks ultimately resulted in the bailouts of Fannie Mae and Freddie Mac by the US government, suggesting that welfare would have been improved without these programs, or if the US Treasury had coordinated the debt sizes of the Agencies along with that of the rest of the federal government. In Europe, the expansion of sovereign debt after the formation of the Euro can similarly be seen as a rat race to serve as the safe asset within Europe. This rat race has also ended badly.

The figure offers a further prediction which may be comforting in today's world of high US government debt. Suppose that the world economy was in region A prior to the financial crisis, but has since transitioned to region B, where US government debt is the top dog. This transition is in keeping with the fact that the amount of US Treasury competitors has fallen since the crisis. The model suggests that the US then has an incentive to shrink its debt.

[10]For natural sizes beyond $\overline{S}(z)$, i.e. $\max(S_1, S_2) > \overline{S}(z)$, there is a region (not shown in Figure 5) outside A that covers the 45-degree line, in which both countries have incentives to shrink. There, the debt sizes are relative large compared to the aggregate funding conditions, and the rollover risk concerns drive both countries to issue less debt.

6 Conclusion

US government debt is the world's premier safe asset currently because i) the US has good fundamentals *relative* to other possible safe assets, and ii) given that global demand for safe assets is currently high, the large float of US government debt is the best parking spot for all of this safe asset demand. In short, there is nowhere else to go. We also derive endogenously the negative β, apparent in a flight to quality, of US government debt. Our analysis of endogenous asset safety also suggest that there can be gains from coordination, and that Eurobonds can exploit these gains by coordinating a security design across Europe.

Our analysis can be extended in other directions. We have taken debt size as well as fundamentals as fixed. But if there is a payoff for a country to ensure that its debt is viewed by investors as a safe asset, then a country is likely to make decisions to capture this payoff. Our investigations of this issue have turned up two results. When countries are roughly symmetric and when global demand for safe assets is high, countries will engage in a rat race to capture a safety premium. Starting from a given, smaller, debt size, and holding fixed the size decision of one country, the other country will have an incentive to increase its debt size since the larger debt size can confer increased safety. But then the first country will have an incentive to respond in a similar way, and so on so forth. In equilibrium, both countries will expand in a self-defeating manner to issue too much debt. The model identifies a second case, when countries are asymmetric and one country is the natural "top dog." In this case, the larger debt country will have an incentive to reduce debts to the point that balances rollover risk and retaining safety, while the smaller country will have an incentive to expand its debt size. Our investigations are suggestive that asymmetry leads to better outcomes than symmetry.

In closing, we emphasize again the main novelty of our analysis of safe assets. Our perspective on safety emphasizes coordination, as opposed to (exclusively) the income process backing the asset, as in conventional analyses of credit risk. In the world, the assets that investors own as their safe assets are largely government debt, money and bank debt. For these assets, valuation has a significant coordination component as in our model, underscoring the relevance of our perspective.

References

[1] **Angeletos, George-Marios, Christian Hellwig, and Alessandro Pavan**, "Signaling in a Global Game: Coordination and Policy Traps," *Journal of Political Economy*, 2006, *114* (3), 452–484.

[2] **——, ——, and ——**, "Dynamic global games of regime change: learning, multiplicity and timing of attacks," *Econometrica*, 2007, *75* (3), 711–756.

[3] **Banerjee, Abhijit V. and Eric S. Maskin**, "A Walrasian Theory of Money and Barter," *Quarterly Journal of Economics*, 1996, *111* (4), 955–1005.

[4] **Bernanke, Ben S.**, "The global savings glut and the US current account deficit," Technical Report, Speech delivered at the Sandbridge Lecture, Virginia Association of Economists, Richmond, VA mar 2005.

[5] **——, Carol Bertaut, Laurie Pounder DeMarco, and Steven Kamin**, "International Capital Flows and the Returns to Safe Assets in the United States, 2003-2007," *Board of Governors of the Federal Reserve System International Finance Discussion Papers*, 2011, *1014*.

[6] **Caballero, Ricardo J and Arvind Krishnamurthy**, "Global Imbalances and Financial Fragility," *American Economic Review*, 2009, *99* (2), 584–588.

[7] **—— and Emmanuel Farhi**, "The Safety Trap," *Working Paper*, 2015.

[8] **——, ——, and Pierre-Olivier Gourinchas**, "An Equilibrium Model of "Global Imbalances" and Low Interest Rates," *American Economic Review*, 2008, *98* (1), 358–393.

[9] **Calvo, Guillermo A.**, "Servicing the Public Debt: The Role of Expectations," *American Economic Review*, 1988, *78* (4), 647–661.

[10] **Carlsson, Hans and Eric van Damme**, "Global Games and Equilibrium Selection," *Econometrica*, 1993, *61* (5), 989–1018.

[11] **Claessens, Stijn, Ashoka Mody, and Shahin Vallee**, "Paths to Eurobonds," *IMF Working Paper*, 2012, *12172*.

[12] **Cole, Harold L and T J Kehoe**, "Self-fulfilling debt crises," *Review of Economic Studies*, 2000, *67*, 91–116.

[13] **Diamond, D W**, "Debt Maturity and Liquidity Risk," *Quarterly Journal of Economics*, aug 1991, *106* (3), 709–737.

[14] **Diamond, Peter A.**, "National Debt in a Neoclassical Growth Model," *American Economic Review*, 1965, *55* (5), 1126–1150.

[15] **Doepke, Matthias and Martin Schneider**, "Money as a Unit of Account," *Working Paper*, 2013.

[16] **Eichengreen, Barry**, "The Euro as a Reserve Currency," *Journal of Japanese and International Economies*, 1998, *12* (4), 483–506.

[17] ——, "The rise and fall of the dollar (or when did the dollar replace sterling as the leading reserve currency?)," *NBER Working Papers*, 2005, *11336.*

[18] ——, *Exorbitant Privilege: the rise and fall of the Dollar and the future of the international monetary system*, Oxford University Press, 2011.

[19] **Frankel, Jeffrey**, "On the dollar," in "The New Palgrave Dictionary of Money and Finance," London: MacMillan Press, 1992.

[20] **Freeman, Scott and Guido Tabellini**, "The Optimality of Nominal Contracts," *Economic Theory*, 1998, *11* (3), 545–562.

[21] **Goldstein, Itay and Ady Pauzner**, "Demand-Deposit Contracts and the Probability of Bank Runs," *Journal of Finance*, jun 2005, *60* (3), 1293–1327.

[22] **Gourinchas, Pierre-Oliver, Helene Rey, and Nicolas Govillot**, "Exorbitant Privilege and Exorbitant Duty," *Working Paper*, 2010.

[23] **Greenwood, Robin and Dimitri Vayanos**, "Bond supply and excess bond returns," *Review of Financial Studies*, 2014, *27* (3), 663–713.

[24] **Hassan, Tarek**, "Country size, currency unions, and international asset returns," *Journal of Finance*, 2013, *68* (6), 2269–2308.

[25] **He, Zhiguo and Konstantin Milbradt**, "Endogenous liquidity and defaultable bonds," *Econometrica*, 2014, *82* (4), 1443–1508.

[26] ___ **and** ___ , "Dynamic debt maturity," *Working Paper*, 2015.

[27] ___ **and Wei Xiong**, "Dynamic debt runs," *Review of Financial Studies*, 2012, *25*, 1799–1843.

[28] ___ **and** ___ , "Rollover Risk and Credit Risk," *Journal of Finance*, 2012, *67* (2), 391–429.

[29] ___ **, Arvind Krishnamurthy, and Konstantin Milbradt**, "What makes US government bonds safe assets?," *American Economic Review Papers & Proceedings*, 2015.

[30] **Holmstrom, Bengt and Jean Tirole**, "Private and Public Supply of Liquidity," *Journal of Political Economy*, feb 1998, *106* (1), 1–40.

[31] **Kiyotaki, Nobuhiro and Randall Wright**, "On Money as a Medium of Exchange," *Journal of Political Economy*, 1989, *97* (4), 927–954.

[32] **Krishnamurthy, Arvind and Annette Vissing-Jorgensen**, "The Aggregate Demand for Treasury Debt," *Journal of Political Economy*, 2012, *120* (2), 233–267.

[33] ___ **and** ___ , "The Impact of Treasury Supply on Financial Sector Lending and Stability," *Working Paper*, 2015.

[34] **Krugman, Paul R.**, "The international role of the dollar: theory and prospect," in "Exchange rate theory and practice," National Bureau of Economic Research, Inc., 1984, chapter NBER Chapt.

[35] **Lagos, Ricardo**, "Asset prices and liquidity in an exchange economy," *Working Paper*, 2005.

[36] **Maggiori, Matteo**, "Financial Intermediation, International Risk Sharing, and Reserve Currencies," *Working Paper*, 2013.

[37] **Morris, Stephen and Hyun S Shin**, "Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks," *American Economic Review*, jun 1998, *88* (3), 587–597.

[38] ___ **and** ___ , "Global Games: Theory and Applications," in Mathias Dewatripont, Lars P Hansen, and S. Turnovsky, eds., *Advances in Economics and Econometrics*, Cambridge University Press, 2003.

[39] ___ **and** ___ , "Illiquidity component of credit risk," *Working Paper*, 2010.

[40] **Rochet, Jean-Charles and Xavier Vives**, "Coordination failure and the lender of last resort: was Bagehot right after all?," *Journal of the European Economic Association*, dec 2004, *2* (6), 1116–1147.

[41] **Samuelson, Paul A.**, "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," *Journal of Political Economy*, 1958, *66* (6), 467–482.

[42] **Vayanos, Dimitri and Pierre-Olivier Weill**, "A Search-Based Theory of the On-the-Run Phenomenon," *Journal of Finance*, 2008, *53* (3), 1361–1398.

A Main Appendix

A.1 Equilibrium with non-monotone strategies and zero recovery

We now construct an equilibrium with non-monotone strategies and joint safety on the endogenously determined interval $[\delta_L, \delta_H]$. Given this equilibrium, we will compute the minimum value of $z = \underline{z}$ for which this equilibrium exists. The possibility of joint safety means that our equilibrium construction using threshold strategies is no longer possible. In a region where both countries are known to be safe (recall we consider the limit where $\sigma \to 0$), investors must be indifferent between the two countries, thus equalizing bond returns. Outside the joint safety interval, i.e., $\tilde{\delta} \in [-\bar{\delta}, \delta_L) \cup (\delta_H, \bar{\delta}]$, we are back to the case where the signal is so strong that only one country is safe.

We conjecture the following non-monotone strategy whereby investment in country 1 and in country 2 alternates on discrete intervals of length $k\sigma$ and $(2-k)\,\sigma$, with $k \in (0,2)$. The investor j's strategy given his private signal δ_j is $\phi(\delta_j) \in \{0,1\}$:

$$\phi(\delta_j) = \begin{cases} 0, & \delta_j < \delta_L \\ 1, & \delta_j \in [\delta_L, \delta_L + k\sigma] \cup [\delta_L + 2\sigma, \delta_L + (2+k)\,\sigma] \cup [\delta_L + 4\sigma, \delta_L + (4+k)\,\sigma] \cup ... \\ 0, & \delta_j \in [\delta_L + k\sigma, \delta_L + 2\sigma] \cup [\delta_L + (2+k)\,\sigma, \delta_L + 4\sigma] \cup [\delta_L + (4+k)\,\sigma, \delta_L + 6\sigma] \cup ... \\ 1, & \delta_j > \delta_H \end{cases} \tag{A.1}$$

As we will show shortly, the non-monotone oscillation occurs only when both countries are safe, where the equilibrium requires proportional investment in each safe country to equalize returns across two safe bonds. Clearly, k determines the fraction of agents in investing in country 1 when oscillation occurs, to which we turn next.

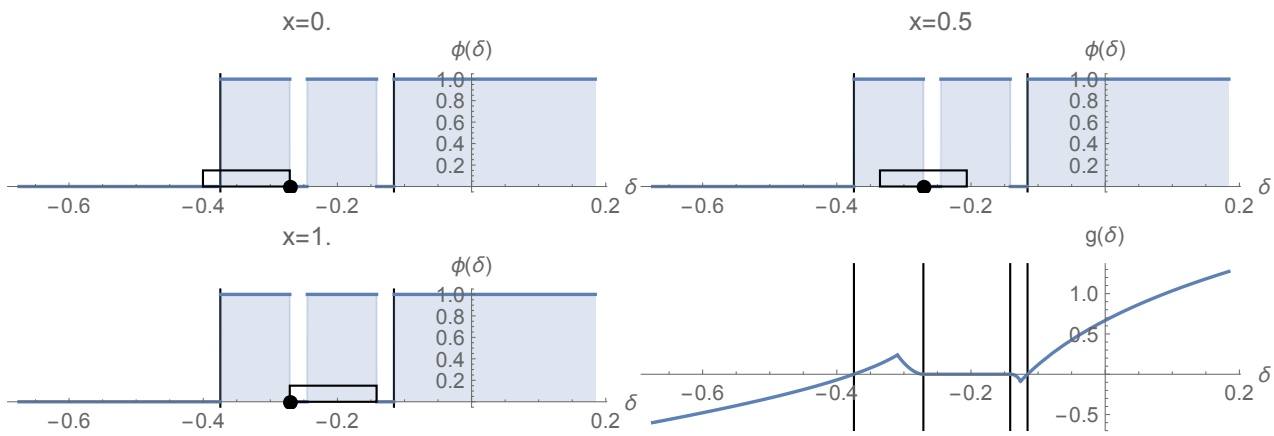

Figure A.1: Investor in interior of joint safety region (black dot) considers range of other investors signals and their strategies (black box) if he has the highest signal ($x = 0$) [top left], median signal ($x = \frac{1}{2}$) [top right], lowest signal ($x = 1$) [bottom left]; expected payoff $g(\delta)$ of investing in country 1 over country 2 as a function of the signal δ_j [bottom right].

Graphical intuition of the proof. Figure A.1 supplies the intuition of the proof, for $\sigma > 0$. Figure A.1 shows an investor (black dot) with a signal deep inside the joint safety region, $\delta_j = \delta_L + k\sigma$. Regardless of his relative position x, this investors knows that the proportions of investors in country 1 and 2 remain constant throughout, leading to joint safety. Thus, the investor is indifferent as the bottom right panel shows, and follows the prescribed equilibrium oscillating strategy. Consider instead an investors on the edge of the joint safety region, $\delta_j = \delta_L$. As $0 < \sigma < \delta_H - \delta_L$, this investor knows for sure that country 2 will survive, regardless of x, but is uncertain if country 1 will survive (it survives for high x, but not for low x). To make this investor indifferent, the total amount of investment cannot be invariant to x in contrast to Figure A.1: to balance the returns when country 1 does not survive (low x), it needs high returns when it does (high x). The highest returns, of course, are achieved when a country just survives, and thus the funding must change as a function of x to give indifference. However, as the signal of the agent in question increases, country 1 safety increases faster than its return drop, leading to (for $\sigma > 0$) a strict incentive to invest in country 1, as shown by the $g(\delta)$ function pushing above 0 for an interval $(\delta_L, \delta_L + k\sigma)$.

A.1.1 Fraction of agents in investing in country 1

Consider a region where all investors know that both countries are safe. In this case, the total investment in country 1 and 2 has to be $\frac{1+f}{1+s}$ and $\frac{s(1+f)}{1+s}$, respectively, to equalize returns. Take an agent with signal δ; introduce the function $\rho(\delta)$, which is the expected proportion of agents investing in country 1 given (own) signal δ. Then, given the assumed strategy for all agents and given that we are in the region where both countries are safe,

$$\rho(\delta) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{\phi(y)}{2\sigma} dy = \frac{k\sigma}{2\sigma}.$$

We choose k so that $\rho(\delta) = \frac{1}{1+s} \iff k = \frac{2}{1+s}$. This is because in equilibrium the proportion investing in country 1 must be constant and equal to $\frac{1}{1+s}$ to equalize returns.

Recall that x denotes the fraction of agents with signal realizations above the agent's private signal δ; and x follows a uniform distribution on $[0, 1]$. For any value of δ and x,

$$\rho(\delta, x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{\phi(y)}{2\sigma} dy = \begin{cases} 0, & \delta + 2\sigma x < \delta_L \\ \frac{\delta+2\sigma x-\delta_L}{2\sigma}, & \delta + 2\sigma x \in (\delta_L, \delta_L + k\sigma) \\ \frac{1}{1+s}, & \delta_H - (2-k)\sigma > \delta > \delta_L + k\sigma \end{cases} \tag{A.2}$$

When we evaluate δ at the marginal agent with signal $\delta = \delta_L$, we have

$$\rho(\delta_L, x) = \begin{cases} 0, & x = 0 \\ x, & x \in \left(0, \frac{1}{1+s}\right) \\ \frac{1}{1+s}, & x > \frac{1}{1+s} \end{cases} \tag{A.3}$$

where we observe that $\rho(\delta_L, x)$ is less than or equal to $\frac{1}{1+s}$.

A.1.2 Lower boundary δ_L

Let $V_\phi(\delta)$ be the expected payoff of strategy $\phi \in \{0, 1\}$ when given signal δ. In the completely safe region discussed above (for δ exceeding δ_L sufficiently), investors were indifferent between both strategies. This is not the case for agent with signals around the threshold signal δ_L: as the agent knows investors with signal below are always investing in country 2, country 1 is a perceived default risk. We now calculate the return of investing in either country, from the perspective of the boundary agent δ_L.

For the boundary agent δ_L, the return from investing only in country 2 (i.e. $\phi = 0$) is given by

$$\Pi_2(\delta_L) = \int_0^1 \frac{s}{(1+f)(1-\rho(\delta_L, x))} dx \tag{A.4}$$

where we integrate over all x as country 2 is safe regardless of x. We will show consistency of this assumption with the derived equilibrium later. Thus, plugging in, we have

$$\Pi_2(\delta_L) = \frac{s}{1+f}\left[\int_0^{\frac{1}{1+s}} \frac{1}{1-x} dx + \int_{\frac{1}{1+s}}^1 \frac{1}{\frac{s}{1+s}} dx\right] = \frac{s}{1+f}\left[\ln\frac{1+s}{s} + 1\right] < \frac{1+s}{1+f}. \tag{A.5}$$

where we used $s \ln \frac{1+s}{s} < 1$. Here, we see that payoff to investing in country 2 is lower than the expected payoff that would have realized if both countries were safe. This reflects the strategic substitution effect: because more people (in expectation) invest in the safe country 2, the return in country 2 is lower.

Now we turn to country 1. Since country 1 has default risk, we need to calculate the threshold $x = x_{min}$ so that country 1 becomes safe if there are $x > x_{min}$ measure of agents receiving better signals. To derive x_{min}, we first solve for $\rho_1^{min}(\delta)$, which is the minimum proportion of agents investing in country 1 that are needed to make country 1 safe given fundamental δ. We have

$$\theta_1(\delta) + (1+f)\rho_1^{min}(\delta) = 1 \iff \rho_1^{min}(\delta) = \frac{1-\theta_1(\delta)}{1+f}$$

Define x_{min} as the solution to $\rho(\delta_L, x) = \rho_1^{min}(\delta_L)$. Given equation (A.3), we have that,

$$x_{min} = \frac{1-\theta_1(\delta_L)}{1+f}. \tag{A.6}$$

The expected return of investing in country 1 given one's own signal δ_L and the conjectured strategies $\phi(\cdot)$ of everyone else is given by,

$$\begin{aligned} \Pi_1(\delta_L) &= \int_{x_{min}}^1 \frac{1}{(1+f)\rho(\delta_L, x)} dx = \frac{1}{1+f}\left[\int_{x_{\min}}^{\frac{1}{1+s}} \frac{1}{x} dx + \int_{\frac{1}{1+s}}^1 \frac{1}{1/(1+s)} dx\right] \\ &= \frac{1}{1+f}\left[\ln\frac{1}{1+s} - \ln x_{min} + s\right]. \end{aligned} \tag{A.7}$$

The boundary agent δ_L must be indifferent between investing in either country, i.e., $\Pi_2(\delta_L) = \Pi_1(\delta_L)$. Plugging

in (A.4) and (A.7), we have

$$\frac{s}{1+f}\left[\ln\frac{1+s}{s}+1\right] = \frac{1}{1+f}\left[\ln\frac{1}{1+s}-\ln x_{min}+s\right] \iff x_{min}=\frac{s^s}{(1+s)^{1+s}}. \tag{A.8}$$

We combine our two equations for x_{min}, (A.6) and (A.8), and use $1-\theta_1(\delta_L)=(1-\theta)\exp(-\delta_L)$, to obtain:

$$\frac{s^s}{(1+s)^{1+s}}=\frac{(1-\theta)\exp(-\delta_L)}{1+f}.$$

Recall $z=\ln\frac{1+f}{1-\theta}$; we have

$$\delta_L(z)=-z+(1+s)\ln(1+s)-s\ln s \tag{A.9}$$

A.1.3 Upper boundary δ_H

The derivation is symmetric to the above. We have

$$\rho(\delta,x)=\int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x}\frac{\phi(y)}{2\sigma}dy=\begin{cases}\frac{1}{1+s}, & \delta-2\sigma(1-x)<\delta_H-(2-k)\sigma\\ \frac{\delta+2\sigma x-\delta_H}{2\sigma} & \delta-2\sigma(1-x)\in(\delta_H-(2-k)\sigma,\delta_H)\\ 1, & \delta-2\sigma(1-x)>\delta_H\end{cases} \tag{A.10}$$

so that

$$\rho(\delta_H,x)=\begin{cases}\frac{1}{1+s}, & x<\frac{1}{1+s}\\ x, & x\in\left(\frac{1}{1+s},1\right)\\ 1, & x=1\end{cases} \tag{A.11}$$

which yields

$$\Pi_1(\delta_H)=\int_0^1\frac{1}{\left(1+\hat{f}\right)\rho(\delta_H,x)dy}dx=\frac{1}{1+f}\left[\ln(1+s)+1\right]<\frac{1+s}{1+f},$$

where we integrated over all x as country 1 is always safe in the vicinity of δ_H.

The default condition for country 2 is

$$s\theta_2(\delta_H)+(1+f)\left[1-\rho_2^{max}(\delta_H)\right]=s \iff 1-\rho_2^{max}(\delta_H)=s\frac{1-\theta_2(\delta_H)}{1+f}$$

where $\rho_2^{max}(\delta)$ is the maximum amount of agents investing in country 1 so that country 2 does not default. Assume, but later verify, that at δ_H we have $1-\rho_2^{max}(\delta_H)<\frac{s}{1+s}$, that is, country 2 would survive even if less than $\frac{s}{1+s}$ of investors invest in country 2. Define $x_{max}(\delta_H)$ as the solution to $\rho(\delta_H,x_{max})=\rho_2^{max}(\delta_H)$; (A.11) implies that

$$1-x_{max}(\delta_H)=s\frac{1-\theta_2(\delta_H)}{1+f}. \tag{A.12}$$

As a result, the return to country 2 is,

$$\begin{aligned}\Pi_2(\delta_H) &= \int_0^{x_{max}(\delta_H)}\frac{s}{(1+f)(1-\rho(\delta_H,x))dy}dx=\frac{s}{1+\hat{f}}\left[\int_0^{\frac{1}{1+s}}\frac{1}{1-\frac{1}{1+s}}dx+\int_{\frac{1}{1+s}}^{x_{max}(\delta_H)}\frac{1}{1-x}dx\right]\\ &= \frac{s}{1+f}\left[\frac{1}{s}+\ln\frac{s}{1+s}-\ln(1-x_{max}(\delta_H))\right]\end{aligned}$$

Indifference at the boundary agent δ_H requires $\Pi_1(\delta_H)=\Pi_2(\delta_H)$, which yields $1-x_{max}(\delta_H)=\frac{s}{(1+s)^{\frac{1+s}{s}}}$. Combining

this result with (A.12) and $1-\theta_2(\delta_H) = (1-\theta)\exp(\delta_H)$, we solve,

$$\delta_H(z) \quad = \quad z - \frac{1+s}{s}\ln(1+s) \tag{A.13}$$

A.1.4 Verifying the equilibrium

We now verify the interior agents $\delta \in (\delta_L, \delta_H)$ have the appropriate incentives to play the conjectured strategy, and that our assumptions of country 1 (2) is always safe at δ_H (δ_L) are correct. As an investor with signal $\delta = \delta_L$ is indifferent, it is easy to show that agents with $\delta < \delta_L$ find it optimal to invest in country 2. Consider an investor with signal $\delta = \delta_L + k\sigma$ (i.e. let us consider the investors depicted by the black dot in Figure A.1). Regardless of his relative position (as measured by x) in the signal distribution, this agent knows that a proportion $\frac{1}{1+s}$ of investors invest in country 1, thus making it safe for sure. Further, he knows that a proportion $\frac{s}{1+s}$ of investors invest in country 2, also making it safe. Therefore, this agent knows that (i) both countries are completely safe and that (ii) investment flows give arbitrage free prices. He is thus indifferent, and so is every investor with $\delta_L + k\sigma < \delta < \delta_H - (2-k)\sigma$.

Next, we consider an investor with $\delta \in (\delta_L, \delta_L + k\sigma)$. We know that country 2 will always survive, and thus we have

$$\Pi_2(\delta) = \int_0^1 \frac{s}{(1+f)\int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{1-\phi(y)}{2\sigma}dy}dx.$$

Note that for any x with $x \geq -\frac{\delta-\delta_L-k\sigma}{2\sigma}$ we are in the oscillating region; for x below we are in the increasing part. Let $\varepsilon \equiv \frac{\delta-\delta_L}{2\sigma} \in \left(0, \frac{1}{1+s}\right)$ so that so that $\delta = \delta_L + 2\sigma\varepsilon$. Thus, we have

$$1-\rho(\delta,x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{1-\phi(y)}{2\sigma}dy \quad = \quad \begin{cases} 1-\varepsilon-x, & x \in \left(0, \frac{1}{1+s}-\varepsilon\right), \\ \frac{s}{1+s}, & x \in \left(\frac{1}{1+s}-\varepsilon, 1\right). \end{cases} \tag{A.14}$$

Then, we have

$$\Pi_2(\delta) \quad = \quad \frac{s}{1+f}\left[\int_0^{\frac{1}{1+s}-\varepsilon} \frac{1}{1-\varepsilon-x}dx + \int_{\frac{1}{1+s}-\varepsilon}^1 \frac{1}{\frac{s}{1+s}}dx\right] = \Pi_2(\delta_L) + \frac{s\left(\ln(1-\varepsilon) + \frac{1+s}{s}\varepsilon\right)}{1+f}$$

For investment in country 1, we know that, since $\delta > \delta_L$, we have $\rho_1^{min}(\delta) < \rho_1^{min}(\delta_L)$. First, note that

$$\rho(\delta,x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x} \frac{\phi(y)}{2\sigma}dy = \begin{cases} \varepsilon+x, & x \in \left(0, \frac{1}{1+s}-\varepsilon\right) \\ \frac{1}{1+s}, & x \in \left(\frac{1}{1+s}-\varepsilon, 1\right) \end{cases}$$

Let $x_{min}(\delta)$ be the measure of investors with higher signals than δ so that country 1 is safe. Since $\rho_1^{min}(\delta) = \frac{1-\theta_1(\delta)}{1+f}$, $x_{min}(\delta)$ is the lowest $x \in [0,1]$ such that

$$\rho(\delta,x) = \varepsilon + x \geq \rho_1^{min}(\delta).$$

Thus, we have

$$x_{min}(\delta) = x_{min}(\delta_L + 2\sigma\varepsilon) \quad = \quad \max\left\{\frac{1-\theta_1(\delta_L+2\sigma\varepsilon)}{1+f} - \varepsilon, 0\right\}. \tag{A.15}$$

The expected investment return from country 1 is

$$\begin{aligned} \Pi_1(\delta) \quad &= \quad \int_{x:\rho(\delta,x)\geq\rho_1^{min}(\delta)} \frac{1}{(1+f)\int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x}\frac{\phi(y)}{2\sigma}dy}dx \\ &= \quad \Pi_1(\delta_L) + \frac{1}{1+f}\left\{\ln x_{min}(\delta_L) - \ln\left[\varepsilon + x_{min}(\delta_L+2\sigma\varepsilon)\right] + (1+s)\varepsilon\right\} \end{aligned}$$

Thus, to show that $\Pi_1(\delta_L + 2\sigma\varepsilon) \geq \Pi_2(\delta_L + 2\sigma\varepsilon)$, we need to show that the following inequality holds for $\varepsilon \in \left(0, \frac{1}{1+s}\right)$:

$$g_L(\varepsilon) \equiv (1+f)(\Pi_1 - \Pi_2) = \ln x_{min}(\delta_L) - \ln[\varepsilon + x_{min}(\delta_L + 2\sigma\varepsilon)] - s\ln(1-\varepsilon) \geq 0. \tag{A.16}$$

First, by using $\ln x_{min}(\delta_L) = s\ln s - (1+s)\ln(1+s)$ and $x_{min}\left(\delta_L + 2\sigma\frac{1}{1+s}\right) = 0$, we know the above inequality holds with equality at both end points $\varepsilon = 0$ and $\varepsilon = \frac{1}{1+s}$, i.e., $g_L(0) = g_L\left(\frac{1}{1+s}\right) = 0$. Second, it is easy to show that there exists a unique ε^* such that $\frac{1-\theta_1(\delta_L+2\sigma\varepsilon^*)}{1+f} = \varepsilon^*$, at which point (A.15) binds at zero. We further note that at $\varepsilon = 0$ we have $\frac{1-\theta_1(\delta_L)}{1+f} > 0$. Thus, in (A.15) we have $\varepsilon^* > 0$ and for $\varepsilon \in (0, \varepsilon^*)$ we have $x_{min}(\delta) = \frac{1-\theta_1(\delta_L+2\sigma\varepsilon)}{1+f} - \varepsilon > 0$, and for $\varepsilon \in \left[\varepsilon^*, \frac{1}{1+s}\right]$ we have $x_{min}(\delta) = 0$. Plugging in and taking derivative with respect to ε, we have

$$\frac{\partial}{\partial\varepsilon}\ln[\varepsilon + x_{min}(\delta_L + 2\sigma\varepsilon)] = \begin{cases} \frac{-2\sigma\theta_1'(\delta_L+2\sigma\varepsilon)}{1-\theta_1(\delta_L+2\sigma\varepsilon)} & ,\varepsilon \in (0,\varepsilon^*) \\ \frac{1}{\varepsilon} & ,\varepsilon \in \left[\varepsilon^*, \frac{1}{1+s}\right] \end{cases}$$

Then, for (A.16), we have $g_L(\varepsilon)$ first rises and then drops:

$$g_L'(\varepsilon) = \begin{cases} \frac{2\sigma\theta_1'(\delta_L+2\sigma\varepsilon)}{1-\theta_1(\delta_L+2\sigma\varepsilon)} + \frac{s}{1-\varepsilon} > 0 & ,\varepsilon \in (0,\varepsilon^*), \\ -\frac{1}{\varepsilon} + \frac{s}{1-\varepsilon} = \frac{(1+s)\varepsilon - 1}{1-\varepsilon} < 0 & ,\varepsilon \in \left[\varepsilon^*, \frac{1}{1+s}\right]. \end{cases}$$

Combined with $g_L(0) = g_L\left(\frac{1}{1+s}\right) = 0$ we know that $g_L(\varepsilon) > 0, \forall\varepsilon \in \left(0, \frac{1}{1+s}\right)$, i.e., Thus, on $\varepsilon \in \left(0, \frac{1}{1+s}\right)$ the investors strictly want to invest in country 1.

We now consider the investors with $\delta \in (\delta_H - (2-k)\sigma, \delta_H)$. We know that country 1 will always survive, and thus we have

$$\Pi_1(\delta) = \int_0^1 \frac{1}{(1+f)\int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x}\frac{\phi(y)}{2\sigma}dy}dx.$$

Let $\varepsilon \equiv \frac{\delta_H - \delta}{2\sigma} \in \left(0, \frac{s}{1+s}\right)$ so that so that $\delta = \delta_H - 2\sigma\varepsilon$. Thus, we have

$$\rho(\delta, x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x}\frac{\phi(y)}{2\sigma}dy \quad = \quad \begin{cases} \frac{1}{1+s}, & x \in \left(0, \frac{1}{1+s} + \varepsilon\right), \\ x - \varepsilon, & x \in \left(\frac{1}{1+s} + \varepsilon, 1\right). \end{cases} \tag{A.17}$$

Plugging in, we have

$$\Pi_1(\delta) = \frac{1}{1+f}\left[\int_0^{\frac{1}{1+s}+\varepsilon}\frac{1}{\frac{1}{1+s}}dx + \int_{\frac{1}{1+s}+\varepsilon}^1 \frac{1}{x-\varepsilon}dx\right] = \frac{1}{1+f}\left[1 + (1+s)\varepsilon + \ln(1-\varepsilon) + \ln(1+s)\right].$$

For investment in country 2, we know that, since $\delta < \delta_H$, we have $1 - \rho_2^{max}(\delta) < 1 - \rho_2^{max}(\delta_L) \iff \rho_2^{max}(\delta_L) < \rho_2^{max}(\delta)$. First, note that

$$1 - \rho(\delta, x) = \int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x}\frac{1-\phi(y)}{2\sigma}dy = \begin{cases} \frac{s}{1+s}, & x \in \left(0, \frac{1}{1+s} + \varepsilon\right), \\ 1 + \varepsilon - x, & x \in \left(\frac{1}{1+s} + \varepsilon, 1\right). \end{cases}$$

Let $x_{max}(\delta)$ be the measure of investors with higher signals than δ so that country 2 is safe. Since $1 - \rho_2^{max}(\delta) = s\frac{1-\theta_2(\delta)}{1+f}$, $x_{max}(\delta)$ is the highest $x \in [0,1]$ such that

$$1 - \rho(\delta, x) = 1 + \varepsilon - x \leq 1 - \rho_2^{max}(\delta).$$

Thus, we have

$$x_{max}(\delta) = x_{max}(\delta_H - 2\sigma\varepsilon) \quad = \quad \min\left\{1+\varepsilon - s\frac{1-\theta_2(\delta_H - 2\sigma\varepsilon)}{1+f}, 1\right\}. \tag{A.18}$$

The expected investment return from country 2 is

$$\begin{aligned}\Pi_2(\delta) &= \int_{x:\rho(\delta,x)\leq\rho_2^{max}(\delta)} \frac{s}{(1+f)\int_{\delta-2\sigma(1-x)}^{\delta+2\sigma x}\frac{1-\phi(y)}{2\sigma}dy}dx \\ &= \frac{s}{1+f}\left[\frac{1+s}{s}\left(\frac{1}{1+s}+\varepsilon\right) - \ln\left[1+\varepsilon - x_{max}(\delta)\right] + \ln\left(\frac{s}{1+s}\right)\right]\end{aligned}$$

Differencing, we have

$$g_H(\varepsilon) = (1+f)\left[\Pi_1(\varepsilon) - \Pi_2(\varepsilon)\right] = \ln(1-\varepsilon) - s\ln s + (1+s)\ln(1+s) + s\ln\left[1+\varepsilon - x_{max}(\delta)\right]$$

with similar properties to $g_L(\varepsilon)$.

Finally, we need to pick σ appropriately so that there exists some natural number $N > 1$ so that $2N\sigma = \delta_H - \delta_L$. For this particular choice of $\sigma = \hat{\sigma}$, the limiting case of zero signal noise can be achieved when we take the sequence of $\sigma_n = \hat{\sigma}/n$ for $n = 1, 2, ...$

A.1.5 Equilibrium properties

First, with joint safety, the probability of survival for country 1 (or the probability of its bonds being the safe asset) is no longer one minus the probability of survival of country 2. Using $\tilde{\delta} \sim \mathbb{U}\left(-\overline{\delta}, \overline{\delta}\right)$, the probability of country 1 survival is

$$\Pr(\text{country 1 safe}) = \frac{\overline{\delta} - \delta_L}{2\overline{\delta}} = \frac{\overline{\delta} + z - (1+s)\ln(1+s) + s\ln s}{2\overline{\delta}}, \tag{A.19}$$

and the probability of country 2 survival is

$$\Pr(\text{country 2 safe}) = \frac{\delta_H + \overline{\delta}}{2\overline{\delta}} = \frac{\overline{\delta} + z - \frac{1+s}{s}\ln(1+s)}{2\overline{\delta}}.$$

As a result, the bonds issued by country 1 are more likely to be the safe assets than that issued by country 2 if the following condition holds:

$$s\ln s - (1+s)\ln(1+s) + \frac{1+s}{s}\ln(1+s) = s\ln s + \left(\frac{1}{s} - s\right)\ln(1+s) > 0. \tag{A.20}$$

This condition always holds: Define $F(s) \equiv s^2\ln s + \left(1-s^2\right)\ln(1+s)$, then $F(s) > 0$ holds for $s \in (0,1)$. It is clear that $F(0) = 0$ while $F(1) = 0$. Simple algebra shows that

$$F'(s) = 2s\ln s - 2s\ln(1+s) + 1, \frac{1}{2}F''(s) = \ln s - \ln(1+s) + 1 - \frac{s}{1+s} = \ln\left(\frac{s}{1+s}\right) + 1 - \frac{s}{1+s}.$$

Let $y = \frac{s}{1+s} <\in (0,1)$; then because it is easy to show $\ln y + 1 - y < 0$ (due to concavity of $\ln y$), we know that $F''(s) < 0$. As a result, $F(s)$ is concave but $F(0) = F(1) = 0$. This immediately implies that $F(s) > 0$, which is our desired result. The condition is the same if we focus on sole survivals only instead of sole and joint survival, i.e., the bonds of country j are the only safe asset, the condition is exactly the same.

Country 1 has the highest likelihood of survival when $s \to 0$, which immediately follow from $-(1+s)\ln(1+s) + s\ln s$ is decreasing in s.

Obviously, the above equilibrium construction requires that $\delta_L(z) < \delta_H(z)$. Since $\delta_L(z)$ in (A.9) is decreasing in z while $\delta_H(z)$ in (A.13) is increasing in z, this condition $\delta_L(z) < \delta_H(z)$ holds if $z > \underline{z}$ so that $\delta_L(\underline{z}) = \delta_H(\underline{z})$

which gives $\underline{z}$:

$$-\underline{z}+(1+s)\ln(1+s)-s\ln s=\underline{z}-\frac{1+s}{s}\ln(1+s)\Rightarrow\underline{z}=\frac{1}{2}\left[\left(2+s+\frac{1}{s}\right)\ln(1+s)-s\ln s\right]$$

A.2 Extension for a negative β asset

Suppose that θ, which proxies for the aggregate fundamental for both countries, is subject to shocks. For convenience, suppose that $\tilde{\theta}$ is drawn from the following uniform distribution $\tilde{\theta}\sim U\left[\underline{\theta},\overline{\theta}\right]$, and recall $z\left(\tilde{\theta}\right)=\ln\frac{1+f}{1-\tilde{\theta}}$. Also, suppose that

$$l_i=l\tilde{\theta}, i\in\{1,2\}$$

where $l>0$ is a positive constant, so that recovery is increasing in the fundamental shock. Using (15), we calculate the threshold $\delta^*(\theta)$ as a function of the realization of $\tilde{\theta}=\theta$, to be

$$\delta^*(\theta)=\frac{[(1-l\theta)s-(1-l\theta)]z(\theta)-(s+l\theta)\ln(s+l\theta)+(1+sl\theta)\ln(1+l\theta s)+l\theta\ln(l\theta)-sl\theta\ln(l\theta)}{(1-l\theta)+s(1-l\theta)}$$

Note that $\frac{d}{d\theta}\delta^*(\theta)<0$; that is, a higher θ, by reducing rollover risk, makes country 1 safer.

In this exercise we consider a distribution so that the relative fundamental δ is almost surely, $\delta>\delta^*(\mathbb{E}[\theta])$. This implies that ex-ante country 1 bonds are more likely to be safe. Also, define $\hat{\theta}(\delta)$ so that $\delta^*\left(\hat{\theta}\right)=\delta$ holds; this is the critical value of fundamental $\theta=\hat{\theta}$ so that country 1's bonds lose safety. We choose δ so that $\hat{\theta}>\underline{\theta}$, which implies that with strictly positive probability, country 1 defaults given a sufficiently low fundamental.

We are interested in the β of the bond price of each country with respect to the θ shock, i.e.,

$$\beta_i(\delta)=\frac{Cov\left(p_i\left(\tilde{\theta};\delta\right),\tilde{\theta}\right)}{Var\left(\tilde{\theta}\right)}=\frac{\mathbb{E}\left[p_i\left(\tilde{\theta};\delta\right)\cdot\tilde{\theta}\right]-\mathbb{E}\left[\tilde{\theta}\right]\mathbb{E}\left[p_i\left(\tilde{\theta};\delta\right)\right]}{Var\left(\tilde{\theta}\right)}, \tag{A.21}$$

From equation (17), we know that

$$p_1(\theta;\delta)=\begin{cases}\frac{(1+f)l\theta}{s+l\theta} & \text{if } \theta<\hat{\theta}(\delta) \text{ so country 1 defaults;}\\ \frac{1+f}{1+l\theta s} & \text{if } \theta\geq\hat{\theta}(\delta) \text{ so country 1 survives;}\end{cases}$$

and

$$p_2(\theta;\delta)=\begin{cases}\frac{1+f}{s+l\theta} & \text{if } \theta<\hat{\theta}(\delta) \text{ so country 2 survives;}\\ \frac{(1+f)l\theta}{1+l\theta s} & \text{if } \theta\geq\hat{\theta}(\delta) \text{ so country 2 defaults.}\end{cases}$$

Given these pricing functions, it is straightforward to evaluate βs in (A.21). We vary country 1's relative strength δ and plot the βs for both bonds as a function of δ in Figure 3. We only plot the β for country 1's bonds, because $\beta_2=-\beta_1/s$ in our model.[11]

A.3 Equilibrium with monotone strategies and joint bonds

In this appendix, we proof that $\delta^*(\alpha)$ is unique, $\delta^*(\alpha)\leq 0$, exists on $[0,\alpha^*]$ and has $\delta^*(\alpha^*)=0$.

First, assume $s=1$. Then, conjecture that $\delta^*(\alpha)=0$ throughout by a simple symmetry argument. From (25), with $\theta_{def}(\delta^*(\alpha))=\theta$, we then have

$$\alpha^*=\frac{1+s}{1+f}(1-\theta)=e^{-z}(1+s) \tag{A.22}$$

Next, assume $s<1$ and $e^z>(1+s)$ so that $\delta^*(0)<0$. Then, let us conjecture $\delta^*(\alpha)\leq 0$ for $\alpha\in(0,\alpha^*)$.

[11] This is because cash-in-the-market-pricing implies that $p_1+sp_2=1+f$.

Setting $\Pi_1(\delta^*) = \Pi_2(\delta^*)$ from (20) after substituting in for $\hat{f}$ from (23), $\delta^*(\alpha)$ is implicitly defined via

$$0 = h(\delta^*, \alpha) \quad = \quad \ln\left[e^z \frac{1-\alpha}{e^{-\delta^*} - \frac{\alpha}{1+s}e^z}\right] - s\ln\left[\frac{e^z}{s}\frac{1-\alpha}{e^{\delta^*} - \frac{\alpha}{1+s}e^z}\right] \tag{A.23}$$

Then, consider $\tilde{\delta} = \delta^*(\alpha)_+$. At this point, country 1 just survives, even though the funding gap (scaled by size) of country 2 is the best among all defaulting countries. Then, for the monotone cutoff strategy to be consistent, we need the default condition

$$\alpha \leq \frac{1+s}{1+f}\left[1 - \theta_2(\delta^*)\right] = \frac{1+s}{1+f}(1-\theta)e^{\delta^*} = e^{-z}(1+s)e^{\delta^*}$$

Suppose that the constraint is binding, which defines a loosest $\delta^*(\alpha)$ by

$$\hat{\delta}^*(\alpha) = z + \ln\left(\frac{\alpha}{1+s}\right) \iff e^{\hat{\delta}^*(\alpha)} = \frac{\alpha}{1+s}e^z \tag{A.24}$$

Assume that $\alpha < \alpha^* = \frac{1+s}{1+f}(1-\theta)$. Plugging in $\hat{\delta}^*(\alpha)$, we see that

$$h\left(\hat{\delta}^*(\alpha), \alpha\right) = \ln\left[e^z \frac{1-\alpha}{e^{-z}\frac{1+s}{\alpha} - \frac{\alpha}{1+s}e^z}\right] - s\ln\left[\frac{e^z}{s}\frac{1-\alpha}{e^z\frac{\alpha}{1+s} - \frac{\alpha}{1+s}e^z}\right] < 0 \tag{A.25}$$

as the second term explodes, i.e. $\ln[\cdot] = \infty$. Thus, it must be that $0 > \delta^*(\alpha) > \hat{\delta}^*(\alpha)$—the first part by our assumption that $\delta^* < 0$ and the second by the construction. However, we note that $\hat{\delta}^*(\alpha^*) = 0$ so that $\delta^*(\alpha^*) = 0$. This is possible as $(\delta, \alpha) = (0, \alpha^*)$ is a root of h – both sides are exploding at this point. The restriction above also implies that $0 < \delta^*_\alpha(\alpha^*) < \hat{\delta}^*_\alpha(\alpha^*) = \frac{1}{\alpha^*}$ so that $\delta^*(\alpha)$ has a bounded and positive derivative at α^*.

We next show that for a fixed $\alpha \in [0, \alpha^*]$, there exists unique $\delta^*(\alpha)$ that solves $h(\delta^*, \alpha)$. Fix α. Then, consider $h(\delta^*, \alpha)$ as a function of δ^*. Differentiating w.r.t. δ^*, we have

$$\frac{\partial h(\delta^*, \alpha)}{\partial \delta^*} \quad = \quad \frac{e^{-\delta^*}\left(e^{\delta^*} - \frac{\alpha}{1+s}e^z\right) + se^{\delta^*}\left(e^{-\delta^*} - \frac{\alpha}{1+s}e^z\right)}{\left(e^{-\delta^*} - \frac{\alpha}{1+s}e^z\right)\left(e^{\delta^*} - \frac{\alpha}{1+s}e^z\right)}$$

Then, given that we have $\alpha < \alpha^*$ and $\hat{\delta}^*(\alpha) < \delta^* < 0$ by assumption, we have

$$\left(e^{-\delta^*} - \frac{\alpha}{1+s}e^z\right) > \left(e^{-\delta^*} - \frac{\alpha^*}{1+s}e^z\right) = e^{-\delta^*} - 1 > 0$$

by assumption on the sign of δ^*. Next, we have

$$\left(e^{\delta^*} - \frac{\alpha}{1+s}e^z\right) > \left(e^{\hat{\delta}^*(\alpha)} - \frac{\alpha}{1+s}e^z\right) = \frac{\alpha}{1+s}e^z - \frac{\alpha}{1+s}e^z = 0$$

by the assumption on $\delta^* \in \left(\hat{\delta}^*(\alpha), 0\right)$. Thus, we have $\frac{\partial h(\delta^*, \alpha)}{\partial \delta^*} > 0$. Finally, we know that $h\left(\hat{\delta}^*(\alpha), \alpha\right) < 0 < h(0, \alpha)$, so that a unique $\delta^*(\alpha) \in \left(\hat{\delta}^*(\alpha), 0\right)$ exists.

What remains to be shown is that $\delta^*(\alpha)$ does not cross 0 before α^*. Suppose it does. Then, there exists an $\hat{\alpha} > 0$ but $\hat{\alpha} \neq \alpha^*$ such that $\delta^*(\hat{\alpha}) = 0$. Then, we have

$$h(0, \hat{\alpha}) = \ln\left[e^z \frac{1-\hat{\alpha}}{1 - \frac{\hat{\alpha}}{1+s}e^z}\right] - s\ln\left[\frac{e^z}{s}\frac{1-\hat{\alpha}}{1 - \frac{\hat{\alpha}}{1+s}e^z}\right] = (1-s)\ln\left[e^z \frac{1-\hat{\alpha}}{1 - \frac{\hat{\alpha}}{1+s}e^z}\right] + s\ln s$$

Setting this equal to 0, we have

$$\ln\left[\frac{1-\hat{\alpha}}{1-\frac{\hat{\alpha}}{1+s}e^{z}}\right]=\frac{-s\ln s}{1+s}-z \iff \frac{1-e^{\left[\frac{-s\ln s}{1+s}-z\right]}}{\left[1-\frac{1}{1+s}e^{\left[\frac{-s\ln s}{1+s}\right]}\right]}=\hat{\alpha}$$

Simplifying, we have

$$\hat{\alpha}=\frac{(1+s)\left(1-s^{\frac{-s}{1+s}}e^{-z}\right)}{1+s-s^{\frac{-s}{1+s}}}$$

Then, notice that $\hat{\alpha}>\alpha^{*} \iff \frac{(1+s)\left(1-s^{\frac{-s}{1+s}}e^{-z}\right)}{1+s-s^{\frac{-s}{1+s}}}>e^{-z}(1+s)$, which simplifies to $1>\alpha^{*}$. Thus, the function $\delta^{*}(\alpha)$ does not cross 0 before α^{*}.

A.4 Equilibrium with non-monotone strategies and joint bonds

Let us conjecture a non-monotone oscillating strategy as in A.1.

A.4.1 Lower boundary δ_L.

The definitions of $\rho(\delta,x)$ and $\rho(\delta_L,x)$ are as in Appendix A.1, and most of the result simply have $\hat{f}$ instead of f: as country 2 is safe to an agent with $\delta=\delta_L$, we have $\Pi_2(\delta_L)=\frac{s}{1+\hat{f}}\left[\ln\frac{1+s}{s}+1\right]<\frac{1+s}{1+\hat{f}}$.

The common bonds change the safety condition for country 1 to

$$\theta_1(\delta)+\alpha p_c+\left(1+\hat{f}\right)\rho_1^{min}(\delta)=1 \iff \rho_1^{min}(\delta)=\frac{1-\theta_1(\delta)-\alpha p_c}{1+\hat{f}}$$

Define $x_{min}(\delta_L)$ as the solution to $\rho(\delta_L,x)=\rho_1^{min}(\delta_L)$. Given equation (A.3), we have that,

$$x_{min}(\delta_L)=\frac{1-\theta_1(\delta_L)-\alpha p_c}{1+\hat{f}} \tag{A.26}$$

Again, the expected return of investing in country 1 is given by $\Pi_1(\delta_L)=\frac{1}{1+\hat{f}}\left[\ln\frac{1}{1+s}-\ln x_{min}(\delta_L)+s\right]$. Indifference requires that $\Pi_2(\delta_L)=\Pi_1(\delta_L)$, which implies that

$$x_{min}(\delta_L) \quad = \quad \exp\left[s\ln s-(1+s)\ln(1+s)\right] \tag{A.27}$$

We combine the expressions for $x_{min}(\delta_L)$, (A.26) and (A.27), to solve for δ_L:

$$\delta_L=-\ln\left\{\frac{1}{1-\theta}\left[\left(1+\hat{f}\right)\frac{s^{s}}{(1+s)^{(1+s)}}+\alpha p_c\right]\right\}. \tag{A.28}$$

A.4.2 Upper boundary δ_H.

The derivation of $\rho(\delta,x)$ and $\rho(\delta_H,x)$ follow Appendix A.1, , and most of the result simply have $\hat{f}$ instead of f. We have $\Pi_1(\delta_H)=\frac{\ln(1+s)+1}{1+\hat{f}}$ as country 1 is considered safe at $\delta_j=\delta_H$.

The default condition for country 2 is

$$s\theta_2(\delta)+s\alpha p_c+\left(1+\hat{f}\right)\left[1-\rho_2^{max}(\delta)\right]=s \iff \left[1-\rho_2^{max}(\delta)\right]=s\frac{1-\theta_2(\delta)-\alpha p_c}{1+\hat{f}}$$

where $\rho_2^{max}(\delta)$ is the maximum amount of people investing in country 1 so that country 2 does not default. Define $x_{max}(\delta_H)$ as the solution to $\rho(\delta_H,x_{max})=\rho_2^{max}(\delta_H)$. Given equation (A.11), we have that,

$$1 - x_{max}(\delta_H) = s\frac{1-\theta_2(\delta_H) - \alpha p_c}{1+\hat{f}} \tag{A.29}$$

Then the return to investing in country 2 is again given by $\Pi_2(\delta_H) = \frac{s}{1+\hat{f}}\left[\frac{1}{s} + \ln\frac{s}{1+s} - \ln(1 - x_{max}(\delta_H))\right]$.

Indifference requires $\Pi_1(\delta_H) = \Pi_2(\delta_H)$, which implies that

$$1 - x_{max}(\delta_H) \quad = \quad \frac{s}{(1+s)^{\frac{1+s}{s}}} \tag{A.30}$$

We combine the expressions for $x_{max}(\delta_H)$, (A.29) and (A.30), to solve for δ_H:

$$\delta_H \quad = \quad \ln\left\{\frac{1}{1-\theta}\left[\frac{1+\hat{f}}{(1+s)^{\frac{1+s}{s}}} + \alpha p_c\right]\right\} \tag{A.31}$$

The remainder of the proof, i.e., the verification argument, is exactly the same as in Appendix A.1 and hence omitted here.

A.4.3 Cutoff $\alpha_{HL} < \alpha^*$.

First, the assumption $e^z > (1+s) \iff (1+f) > (1-\theta)(1+s)$ guarantees that there is *some* realizations of $\tilde{\delta}$ that would allow joint safety. Consider the total funding requirement,

$$total\left(\tilde{\delta}\right) = (1-\theta_1) + (1-\theta_2)s = (1-\theta)\left(e^{-\tilde{\delta}} + s\cdot e^{\tilde{\delta}}\right) \tag{A.32}$$

This is minimized at $\tilde{\delta} = -\frac{1}{2}\ln s \geq 0$ for a total funding requirement of $total\left(-\frac{1}{2}\ln s\right) = (1-\theta)2\sqrt{s}$. Next, note that $1+s > 2\sqrt{s}$ so that $e^z > (1+s) > 2\sqrt{s}$.

Recall that $\alpha^* = e^{-z}(1+s)$. Then, assume that $z > \ln(1+s)$ so that $\alpha^* \in (0,1)$. Then, we have

$$\begin{aligned}
\delta_H(\alpha^*) - \delta_L(\alpha^*) \quad &= \quad \ln\left\{\frac{e^z}{1+s}\left[\left(\frac{1}{1+s}\right)^{\frac{1}{s}}(1-\alpha^*) + \alpha^*\right]\right\} + \ln\left\{\frac{e^z}{1+s}\left[\left(\frac{s}{1+s}\right)^{s}(1-\alpha^*) + \alpha^*\right]\right\} \\
&= \quad \ln\left[\left(\frac{1}{1+s}\right)^{\frac{1}{s}}\left(\frac{1}{\alpha^*} - 1\right) + 1\right] + \ln\left[\left(\frac{s}{1+s}\right)^{s}\left(\frac{1}{\alpha^*} - 1\right) + 1\right] > 0
\end{aligned}$$

where we used $\left(\frac{1}{1+s}\right)^{\frac{1}{s}} < 1$ and $\left(\frac{s}{1+s}\right)^{s} < 1$ and $\frac{1}{\alpha^*} > 1$ in the last line. Thus, at α^* the oscillating equilibrium already exists. It is easy to show that the the joint safety region $[\delta_L(\alpha), \delta_H(\alpha)]$ is expanding uniformly in α, and thus that $\alpha_{HL} < \alpha^*$.

Finally, define α_{HL} as the solution to

$$\begin{aligned}
0 \quad &= \quad \delta_H(\alpha_{HL}) - \delta_L(\alpha_{HL}) \\
&= \quad 2[z - \ln(1+s)] + \ln\left[\left(\frac{1}{1+s}\right)^{\frac{1}{s}}(1-\alpha_{HL}) + \alpha_{HL}\right] + \ln\left[\left(\frac{s}{1+s}\right)^{s}(1-\alpha_{HL}) + \alpha_{HL}\right]
\end{aligned}$$

Rearranging, we have

$$\left[\left(\frac{1}{1+s}\right)^{\frac{1}{s}}(1-\alpha_{HL}) + \alpha_{HL}\right]\left[\left(\frac{s}{1+s}\right)^{s}(1-\alpha_{HL}) + \alpha_{HL}\right] - e^{-2z}(1+s)^2 = 0$$

which is a quadratic equation in α_{HL}. We note that $e^{-2z}(1+s)^2 < 1 \iff 2[\ln(1+s) - z] < 0$, so that $\alpha_{HL} = 1$ makes the LHS positive. We also know that the LHS is increasing in α_{HL} for $\alpha_{HL} > 0$. Thus, there exists at most one positive root $\alpha_{HL} \in (0,1)$ under the assumption $z > \ln(1+s)$, and if not, both roots are negative. Solving for

the larger root α_{HL}, and after some algebra, we can show that $\delta^* (\alpha_{HL}) = \delta_H (\alpha_{HL}) = \delta_L (\alpha_{HL})$.

B Online Appendix

B.1 Additive Fundamental Structure

We have considered the specification of $1-\theta_i = (1-\theta)\exp\left((-1)^i\,\tilde{\delta}\right)$ for country i's fundamental. We now show that results are qualitatively similar with the alternative additive specification

$$\theta_1 = \theta + \tilde{\delta}, \text{ and } \theta_2 = \theta - \tilde{\delta}.$$

As $x = \Pr\left(\tilde{\delta} + \epsilon_j > \delta^*\right) = \frac{\tilde{\delta}+\sigma-\delta^*}{2\sigma} \Rightarrow \tilde{\delta} = \delta^* + (2x-1)\,\sigma$, we know that

$$\begin{aligned}\theta_1 &= \theta + \tilde{\delta} = \theta + \delta^* + (2x-1)\,\sigma \\ \theta_2 &= \theta - \tilde{\delta} = \theta - \delta^* - (2x-1)\,\sigma\end{aligned}$$

Given x, the large country 1 survives if and only if

$$p_1 - 1 + \theta_1 = (1+f)\,x - 1 + \theta + \delta^* + (2x-1)\,\sigma \geq 0 \Leftrightarrow x \geq \frac{1-\theta-\delta^*+\sigma}{1+f+2\sigma}$$

which implies the expected return from investing in country 1 is

$$\Pi_1 = \int_{\frac{1-\theta-\delta^*+\sigma}{1+f+2\sigma}}^{1} \frac{1}{(1+f)\,x}dx = \frac{1}{1+f}\ln\frac{1+f+2\sigma}{1-\theta-\delta^*+\sigma}.$$

For country 2, the bond is paid back if

$$\begin{aligned}(1+f)\,x' - s + s\theta_2 &= (1+f)\,x' - s + s\left[\theta - \delta^* - (2x-1)\,\sigma\right] \geq 0 \\ \Leftrightarrow x' &\geq \frac{s\,(1-\theta+\delta^*-\sigma)}{1+f+2s\sigma}\end{aligned}$$

which implies an expected return of

$$\Pi_2 = \int_{\frac{s(1-\theta+\delta^*-\sigma)}{1+f+2s\sigma}}^{1} \frac{s}{(1+f)\,x'}dx' = \frac{s}{1+f}\ln\frac{1+f+2s\sigma}{s\,(1-\theta+\delta^*+\sigma)}$$

As a result, the equilibrium threshold δ^* is pinned by by the indifference condition

$$\ln\frac{1+f+2\sigma}{1-\theta-\delta^*+\sigma} = s\ln\frac{1+f+2s\sigma}{s\,(1-\theta+\delta^*+\sigma)}.$$

Letting $\sigma \to 0$ we obtain

$$\ln\frac{1+f}{1-\theta-\delta^*} = s\ln\frac{1+f}{s\,(1-\theta+\delta^*)}. \tag{B.1}$$

We no longer have close-form solution for δ^* in (B.1), as δ^* shows up in both sides. However, the solution is unique because LHS (RHS) is increasing (decreasing) in δ^*. Finally, to ensure $\delta^* < 0$ so that the larger country 1 is relatively safer, we require the same sufficient condition of $z = \ln\frac{1+f}{1-\theta} > 1$ in this alternative specification.

B.2 Uniqueness of the threshold equilibrium within monotone strategies

First, let us define a few primitives. Let δ_j be a generic signal, and δ be the true state of the world. Further, let x denote the amount of pessimism of the investors, so that $x = 1$ is the most pessimistic agent (amongst all agents out there) and $x = 0$ is the least pessimistic agent. We then have $\delta\left(\delta_j, x\right) = \delta_j + 2\sigma\left(x - \frac{1}{2}\right)$. For most of the proofs, we assume wlog that we the investor believes his signal to be the true signal, and thus all the action comes form movements in his relative position. As $\sigma \to 0$, fundamental uncertainty (that is movements in δ as a function of x)

will vanish, whereas strategic uncertainty (relative ranking of investors as represented by x) remains.

Next, let us define $\phi(\delta_j)$ as the **proportion of funds** an investor with signal δ_j invests in country 1. Then define

$$\rho(\delta_j, x) = \frac{1}{2\sigma}\int_{\delta_j - 2\sigma(1-x)}^{\delta_j + 2\sigma x} \phi(y)\, dy$$

as the **aggregate proportion of investors in country 1** an investor with signal δ_j and level of pessimism x expects given the conjecture strategies $\phi(\cdot)$. Note that there is translation invariance

$$\rho(\delta_j, x) = \rho\left(\delta_j + \varepsilon, x - \frac{\varepsilon}{2\sigma}\right), \forall x \in \left(\frac{\varepsilon}{2\sigma}, 1\right)$$

Finally, define the (scaled by $1+f$) **expected difference in expected returns** as

$$\Delta(\delta_j) = \int_0^1 \mathbf{1}_{\{\rho(\delta,x)\geq\rho_{min}(\delta)\}}\frac{1}{\rho(\delta,x)}dx - \int_0^1 \mathbf{1}_{\{\rho(\delta,x)\leq\rho_{max}(\delta)\}}\frac{s}{1-\rho(\delta,x)}dx$$

Then, for any given conjectured difference function $\Delta(y)$, we must have

$$\phi(y) = \begin{cases} 1, & \Delta(y) > 0 \\ \in [0,1], & \Delta(y) = 0 \\ 0, & \Delta(y) < 0 \end{cases}$$

A **monotone strategy** is defined by $\phi'(y) \geq 0$ for all $y \in \left[-\overline{\delta}, \overline{\delta}\right]$, which implies that $\rho_\delta(\delta, x) \geq 0$ as well as $\rho_x(\delta, x) \geq 0$, i.e., $\rho(\delta, x)$ is monotone. This implies that we can write

$$\begin{aligned}
\Delta(\delta_j) &= \int_0^1 \mathbf{1}_{\{\rho(\delta_j,x)\geq\rho_{min}(\delta(\delta_j,x))\}}\frac{1}{\rho(\delta,x)}dx - \int_0^1 \mathbf{1}_{\{\rho(\delta_j,x)\leq\rho_{max}(\delta(\delta_j,x))\}}\frac{s}{1-\rho(\delta,x)}dx \\
&\approx \int_0^1 \mathbf{1}_{\{\rho(\delta_j,x)\geq\rho_{min}(\delta_j)\}}\frac{1}{\rho(\delta,x)}dx - \int_0^1 \mathbf{1}_{\{\rho(\delta_j,x)\leq\rho_{max}(\delta_j)\}}\frac{s}{1-\rho(\delta,x)}dx \\
&= \int_{x_{min}(\delta_j)}^1 \frac{1}{\rho(\delta_j,x)}dx - \int_0^{x_{max}(\delta_j)} \frac{s}{1-\rho(\delta_j,x)}dx
\end{aligned}$$

Country 1 survives if $\rho(\delta_j, x)$ is larger than $\rho_{min}(\delta(\delta_j, x))$. As the agent becomes more pessimistic relative to the other agents, i.e., x increases, the actual relative fundamental increases, and thus the threshold decreases:

$$\begin{aligned}
\partial_x \rho_{min}(\delta(\delta_j, x)) &= \partial_x e^{-z} e^{-\delta(\delta_j,x)} = -e^{-z}e^{-\delta(\delta_j,x)}2\sigma < 0 \\
\partial_{\delta_j} \rho_{min}(\delta(\delta_j, x)) &= -e^{-z}e^{-\delta(\delta_j,x)} < 0
\end{aligned}$$

Thus, **if $\rho(\delta, x)$ is monotone, there exists a unique threshold** $x_{min}(\delta)$ above which country 1 is safe. Further, by the implicit function theorem, we have

$$\begin{aligned}
x'_{min}(\delta) &= -\frac{\rho_\delta(\delta,x) - \partial_\delta \rho_{min}\left(\tilde{\delta}(\delta,x)\right)}{\rho_x(\delta,x) - \partial_x \rho_{min}\left(\tilde{\delta}(\delta,x)\right)} \\
&= -\frac{\frac{\phi(\delta+2\sigma x)-\phi(\delta-2\sigma(1-x))}{2\sigma} + e^{-z}e^{-\tilde{\delta}(\delta,x)}}{\phi(\delta+2\sigma x) - \phi(\delta - 2\sigma(1-x)) + e^{-z}e^{-\tilde{\delta}(\delta,x)}2\sigma} \\
&= -\frac{1}{2\sigma}
\end{aligned}$$

so that the pessimism threshold falls that makes country 1 safe. Similarly, we have

$$\begin{aligned}
x'_{max}(\delta) &= -\frac{\rho_\delta(\delta,x) - \partial_\delta \rho_{max}\left(\tilde{\delta}(\delta,x)\right)}{\partial_x(\delta,x) - \partial_x \rho_{max}\left(\tilde{\delta}(\delta,x)\right)} \\
&= -\frac{\frac{\phi(\delta+2\sigma x)-\phi(\delta-2\sigma(1-x))}{2\sigma} + se^{-z}e^{\tilde{\delta}(\delta,x)}}{\phi(\delta+2\sigma x)-\phi(\delta-2\sigma(1-x)) + se^{-z}e^{\tilde{\delta}(\delta,x)}2\sigma} \\
&= -\frac{1}{2\sigma}
\end{aligned}$$

We can thus approximate

$$x_{max}(\delta+\varepsilon) + \frac{\varepsilon}{2\sigma} \approx x_{max}(\delta) + x'_{max}(\delta)\varepsilon + \frac{\varepsilon}{2\sigma} = x_{max}(\delta) \quad \text{and} \quad x_{min}(\delta+\varepsilon) + \frac{\varepsilon}{2\sigma} \approx x_{min}(\delta)$$

Finally, suppose a δ exists for which the investor expects **joint safety**, i.e., both countries to be safe for sure. Then, we must have $\phi(\delta) = \frac{1}{1+s}$ by the no arbitrage condition. A threshold equilibrium is defined by a **single-crossing** condition on $\Delta = \Pi_1 - \Pi_2$ and a **non-flat part at** 0, where $\Delta(\delta) > 0$ implies $\phi = 1$ and $\Delta(\delta) < 0$ implies $\phi = 0$. Consider any other equilibrium. By dominance regions, we know that for high δ, $\phi = 1$ will eventually be optimal, and for very low δ, $\phi = 0$ will eventually be optimal.

Thus, any other equilibrium is either characterized by (1) a flat part $\Delta(\delta) = 0$, (2) multiple crossings $\Delta(\delta) = 0$ or (3) a combination of the two. In our oscillating strategy, (3) is the case, with a flat part in the middle.

B.2.1 Monotonicity and uniqueness of threshold equilibrium

A monotone strategy $\phi(\delta)$ requires $\Delta(\delta)$ to change signs only once. Thus, $\Delta(\delta)$ either crosses zero at a single point, or approaches it from below, stays flat on an interval $[\delta_L, \delta_H]$, and then rises above zero. Thus, at any point δ s.t. $\Delta(\delta) = 0$ we must have $\Delta'(\delta) \geq 0$. As we want to show that a threshold equilibrium is the only equilibrium possible, we now rule out any flat parts of Δ at zero.

To this end, suppose an interval $[\delta_L, \delta_H]$ exists on which $\Delta(\delta) = 0$.

Interior x_{min}, x_{max}. Suppose now that $x_{min}(\delta), x_{max}(\delta) \in (0,1)$. This means that both countries are at risk of default, so there is no possibility of joint safety across all possible $x \in [0,1]$ (it might exists for some x if $x_{min}(\delta) < x_{max}(\delta)$). Take $\varepsilon \in (0, \delta_H - \delta_L)$. Then, we write

$$\begin{aligned}
\Pi_1(\delta+\varepsilon) &= \int_{x_{min}(\delta+\varepsilon)}^{1} \frac{1}{\rho(\delta+\varepsilon,x)}dx \\
&= \int_{x_{min}(\delta+\varepsilon)+\frac{\varepsilon}{2\sigma}}^{1+\frac{\varepsilon}{2\sigma}} \frac{1}{\rho\left(\delta+\varepsilon, x-\frac{\varepsilon}{2\sigma}\right)}dx \\
&= \int_{x_{min}(\delta+\varepsilon)+\frac{\varepsilon}{2\sigma}}^{1} \frac{1}{\rho\left(\delta+\varepsilon, x-\frac{\varepsilon}{2\sigma}\right)}dx + \int_{1}^{1+\frac{\varepsilon}{2\sigma}} \frac{1}{\rho\left(\delta+\varepsilon, x-\frac{\varepsilon}{2\sigma}\right)}dx \\
&\approx \int_{x_{min}(\delta)}^{1} \frac{1}{\rho(\delta,x)}dx + \int_{1-\frac{\varepsilon}{2\sigma}}^{1} \frac{1}{\rho(\delta+\varepsilon,x)}dx \\
&= \Pi_1(\delta) + \underbrace{\int_{1-\frac{\varepsilon}{2\sigma}}^{1} \frac{1}{\rho(\delta+\varepsilon,x)}dx}_{\text{new pessimists}}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\Pi_2(\delta+\varepsilon) &= \int_0^{x_{max}(\delta+\varepsilon)} \frac{s}{1-\rho(\delta+\varepsilon,x)}dx \\
&= \int_{\frac{\varepsilon}{2\sigma}}^{x_{max}(\delta+\varepsilon)+\frac{\varepsilon}{2\sigma}} \frac{s}{1-\rho\left(\delta+\varepsilon,x-\frac{\varepsilon}{2\sigma}\right)}dx \\
&\approx \int_{\frac{\varepsilon}{2\sigma}}^{x_{max}(\delta)} \frac{s}{1-\rho(\delta,x)}dx + \int_0^{\frac{\varepsilon}{2\sigma}} \frac{s}{1-\rho(\delta,x)}dx - \int_0^{\frac{\varepsilon}{2\sigma}} \frac{s}{1-\rho(\delta,x)}dx \\
&= \int_0^{x_{max}(\delta)} \frac{s}{1-\rho(\delta,x)}dx - \int_0^{\frac{\varepsilon}{2\sigma}} \frac{s}{1-\rho(\delta,x)}dx \\
&= \Pi_2(\delta) - \underbrace{\int_0^{\frac{\varepsilon}{2\sigma}} \frac{s}{1-\rho(\delta,x)}dx}_{\text{old optimists}}
\end{aligned}$$

so that

$$\begin{aligned}
\Delta(\delta_L+\varepsilon) &= \Pi_1(\delta_L+\varepsilon) - \Pi_2(\delta_L+\varepsilon) \\
&= \Pi_1(\delta_L) + \int_{1-\frac{\varepsilon}{2\sigma}}^{1} \frac{1}{\rho(\delta_L+\varepsilon,x)}dx - \left[\Pi_2(\delta_L) - \int_0^{\frac{\varepsilon}{2\sigma}} \frac{s}{1-\rho(\delta_L,x)}dx\right] \\
&= \int_{1-\frac{\varepsilon}{2\sigma}}^{1} \frac{1}{\rho(\delta_L+\varepsilon,x)}dx + \int_0^{\frac{\varepsilon}{2\sigma}} \frac{s}{1-\rho(\delta_L,x)}dx > 0
\end{aligned}$$

But this implies that

$$\phi(\delta_L+\varepsilon) = 1$$

By monotonicity then, δ_L is the only point at which $\Delta(\delta) = 0$ and no flat parts can exist for $x_{min}, x_{max} \in (0,1)$.

Cornered x_{min}, x_{max}. Next, suppose that at least one of the countries is going to survive regardless of x because of the assumed strategies. Wlog, let us focus on δ_L. First, let us rule out that $x_{min}(\delta_L) = 0$. Note that for any $\varepsilon > 0$, we have by the dominance boundaries $\Delta(\delta_L - \varepsilon) < 0$ and $\Delta(\delta_H + \varepsilon) > 0$, the highest and lowest point of the all flat parts. Further note that $x_{min}(\delta_L) = 0$ implies that country 1 always survives in the eyes of an investor with signal δ_L. By construction we have $\rho(\delta, 0) = 0$— when the agent with signal δ_L is the most optimistic agent, he must believe by the conjecture on $\Delta(\delta)$ that everyone below him investors fully into country 2. But then this agent cannot believe that country 1 is safe regardless of x, as by assumption no country can survive without a minimum amount of investment.

Thus, at δ_L we must have $x_{max}(\delta_L) = 1$ and $x_{min}(\delta_H) = 0$—**country 2 always survives** given the strategies of the different agents. Then, we have the survival boundary of country 2 not changing, and thus again for $\varepsilon \in$

$(0, \delta_H - \delta_L)$ we have

$$\begin{aligned}
\Pi_2(\delta+\varepsilon) &= \int_0^1 \frac{s}{1-\rho(\delta+\varepsilon, x)} dx \\
&= \int_{\frac{\varepsilon}{2\sigma}}^{1+\frac{\varepsilon}{2\sigma}} \frac{s}{1-\rho\left(\delta+\varepsilon, x-\frac{\varepsilon}{2\sigma}\right)} dx \\
&= \int_{\frac{\varepsilon}{2\sigma}}^{1} \frac{s}{1-\rho(\delta, x)} dx + \int_{1-\frac{\varepsilon}{2\sigma}}^{1} \frac{s}{1-\rho(\delta+\varepsilon, x)} dx \\
&= \int_0^1 \frac{s}{1-\rho(\delta, x)} dx + \int_{1-\frac{\varepsilon}{2\sigma}}^{1} \frac{s}{1-\rho(\delta+\varepsilon, x)} dx - \int_0^{\frac{\varepsilon}{2\sigma}} \frac{s}{1-\rho(\delta, x)} dx \\
&= \Pi_2(\delta) + \underbrace{\int_{1-\frac{\varepsilon}{2\sigma}}^{1} \frac{s}{1-\rho(\delta+\varepsilon, x)} dx}_{\text{new pessimists}} - \underbrace{\int_0^{\frac{\varepsilon}{2\sigma}} \frac{s}{1-\rho(\delta, x)} dx}_{\text{old optimists}}
\end{aligned}$$

Then, we have

$$\begin{aligned}
0 = \Delta(\delta+\varepsilon) &= \Pi_1(\delta+\varepsilon) - \Pi_2(\delta+\varepsilon) \\
&= \Pi_1(\delta) + \int_{1-\frac{\varepsilon}{2\sigma}}^{1} \frac{1}{\rho(\delta+\varepsilon, x)} dx - \left[\Pi_2(\delta) + \int_{1-\frac{\varepsilon}{2\sigma}}^{1} \frac{s}{1-\rho(\delta+\varepsilon, x)} dx - \int_0^{\frac{\varepsilon}{2\sigma}} \frac{s}{1-\rho(\delta, x)} dx\right] \\
&= \int_{1-\frac{\varepsilon}{2\sigma}}^{1} \frac{1}{\rho(\delta+\varepsilon, x)} dx - \int_{1-\frac{\varepsilon}{2\sigma}}^{1} \frac{s}{1-\rho(\delta+\varepsilon, x)} dx + \int_0^{\frac{\varepsilon}{2\sigma}} \frac{s}{1-\rho(\delta, x)} dx \\
&= \underbrace{\int_{1-\frac{\varepsilon}{2\sigma}}^{1} \left[\frac{1}{\rho(\delta+\varepsilon, x)} - \frac{s}{1-\rho(\delta+\varepsilon, x)}\right] dx}_{\text{new pessimists}} + \underbrace{\int_0^{\frac{\varepsilon}{2\sigma}} \frac{s}{1-\rho(\delta, x)} dx}_{\text{old optimists}}
\end{aligned}$$

and there is now a *possibility* of a flat part. The intuition here is that we are balancing the returns that arise to the new most pessimistic investor (i.e. for high x) against the previous expected returns of the most optimistic investors (i.e. low x).

Taking derivatives around $\varepsilon = 0$, we have

$$\begin{aligned}
\Delta(\delta+\varepsilon) &\approx \Delta(\delta) + \Delta'(\delta)\varepsilon \\
&= \frac{1}{2\sigma}\left[\frac{1}{\rho\left(\delta+\varepsilon, 1-\frac{\varepsilon}{2\sigma}\right)} - \frac{s}{1-\rho\left(\delta+\varepsilon, 1-\frac{\varepsilon}{2\sigma}\right)}\right]_{\varepsilon=0} \varepsilon \\
&\quad + \left[\int_{1-\frac{\varepsilon}{2\sigma}}^{1} \left[-\frac{\rho_\delta(\delta+\varepsilon, x)}{\rho(\delta+\varepsilon, x)^2} - \frac{s\left(-\rho_\delta(\delta+\varepsilon, x)\right)}{\left[1-\rho(\delta+\varepsilon, x)\right]^2}\right] dx\right]_{\varepsilon=0} \varepsilon \\
&\quad + \frac{1}{2\sigma}\left[\frac{s}{1-\rho\left(\delta, \frac{\varepsilon}{2\sigma}\right)}\right]_{\varepsilon=0} \varepsilon \\
&= \frac{1}{2\sigma}\left[\frac{1}{\rho(\delta, 1)} - \frac{s}{1-\rho(\delta, 1)} + \frac{s}{1-\rho(\delta, 0)}\right] \varepsilon
\end{aligned}$$

When $\delta = \delta_L$ we must have $\rho(\delta_L, 0) = 0$ by definition of δ_L. Then, the derivative $\Delta'(\delta_L) = 0$ if

$$\rho(\delta_L, 1) = \frac{-1+\sqrt{1+4s}}{2s} > \frac{1}{1+s}$$

which implies that least on for some points on $(\delta_L, \delta_L + 2\sigma)$ we have $\phi(\delta) > \frac{1}{1+s}$.

By $x'_{min}(\delta) \leq 0$ and $x'_{max}(\delta) \leq 0$, as δ increases either we (i) move to a segment where $x_{min}(\delta), x_{max}(\delta) \in (0, 1)$,

an interior situation, or (ii) to a segment with $x_{min}(\delta) = 0, x_{max}(\delta) = 1$, a completely safe part.

But we know from the previous section that (i) immediately has $\Delta'(\delta) > 0$, a violation of the premise that we are on a flat part for $\delta \in [\delta_L, \delta_H]$. Next, consider for (ii) any completely safe subset $J \subset (\delta_L, \delta_H)$ and $\delta \in J$. Then, we require $\rho(\delta, x) = \frac{1}{1+s}, \forall x \in [0,1]$ by no arbitrage, which implies $\phi(\delta) = \frac{1}{1+s}$. But then we have a violation of monotonicity as $\rho(\delta_L, 1) > \frac{1}{1+s}$. Thus, there cannot be any flat parts of $\Delta(\delta)$ at zero and the only equilibrium that survives is of the threshold form. By the construction in the paper, this threshold equilibrium is unique.

Existence of threshold equilibrium. Consider our unique candidate equilibrium

$$\delta^* = -\frac{1-s}{1+s}z - \frac{s \ln s}{1+s}$$

derived in the main text. Consider now $\delta_j < \delta^*$. Then, we have

$$\Delta(\delta_j; \delta^*) = \int_{\rho(x) > \rho_{min}\left(\tilde{\delta}(x;\delta_j)\right)} \frac{1}{(1+f)\rho(x)} dx - s \int_{\rho(x) < \rho_{max}\left(\tilde{\delta}(x;\delta_j)\right)} \frac{1}{(1+f)(1-\rho(x))} dx$$

We know that $\Delta(\delta^*; \delta^*) = 0$. But by our setup, we know that moving $\delta_j < \delta^*$ lowers both $\rho_{min}(\delta)$ and $\rho_{max}(\delta)$. Thus, we need to look at the difference between the parts we are adding (region in which country 1 survives) and parts we are subtracting (region in which country 2 survives):

$$\begin{aligned} \Delta_{\delta_j}(\delta_j; \delta^*) &= -\rho'_{min}(\delta_j) \frac{1}{(1+f)\rho_{min}(\delta_j)} + s\rho'_{max}(\delta_j) \frac{1}{(1+f)(1-\rho_{max}(\delta_j))} \\ &= \frac{1}{(1+f)} - s\frac{1}{(1+f)} = \frac{1-s}{1+f} > 0 \end{aligned}$$

where we used

$$\rho'_{min}(\delta_j) = -\rho_{min}(\delta_j) \quad \text{and} \quad \rho'_{max}(\delta_j) = -(1 - \rho_{max}(\delta_j))$$

This is intuitive: as we increase δ_j, we are adding the most valuable states for country 1 (fixing $\rho(x)$) by evaluating at points set on which it will just survive, i.e., close to $\rho_{min}(\delta_j)$, and we are taking away the most valuable states for country 2 by evaluating at points set on which it will just default, i.e., close to $\rho_{max}(\delta_j)$.

B.3 Equilibrium with non-monotone strategies and positive recovery

Let us say that $s_1 = 1, s_2 = s$ and $l_i s_i$ to be the recovery given default of country i, so that it returns $\frac{l_i s_i}{y_i}$ per unit of dollar invested, where y_i is total investment in country i. Then if country 1 survives, to equalize return, we need

$$\frac{l_2 s}{y_2} = \frac{1}{y_1}, y_1 + y_2 = 1 + f \Rightarrow \frac{y_1}{y_2} = \frac{1}{l_2 s}.$$

This gives prices equal to

$$\begin{aligned} p_1 &= y_1 = \frac{(1+f)}{1+l_2 s} \\ p_2 &= \frac{y_2}{s} = \frac{(1+f)\, l_2}{1+l_2 s} \end{aligned}$$

Similarly, if country 2 survives, then

$$\frac{s}{y_2} = \frac{l_1}{y_1}, y_1 + y_2 = 1 + f \Rightarrow \frac{y_1}{y_2} = \frac{l_1}{s}$$

which results in prices

$$\begin{aligned} p_1 &= y_1 = \frac{(1+f)\,l_1}{l_1+s} \\ p_2 &= \frac{y_2}{s} = \frac{(1+f)}{l_1+s} \end{aligned}$$

Let

$$z = \ln \frac{1+f}{1-\theta} > 0$$

and fiscal surplus is given by

$$\begin{aligned} \theta_1 &= 1-(1-\theta)\,e^{-\delta} = 1-(1+f)\,e^{-z}e^{-\delta} \\ s\theta_2 &= s\left[1-(1-\theta)\,e^{\delta}\right] = s\left[1-(1+f)\,e^{-z}e^{\delta}\right] \end{aligned}$$

Define two constants $k_1 > 1$ and $k_2 > 1$ (which only occurs if $s < l_1$) so that

$$\begin{aligned} \frac{k_1}{2-k_1} = \frac{1}{l_2 s} \quad &\Longleftrightarrow \quad k_1 = \frac{2}{1+l_2 s} > 1 \\ \frac{k_2}{2-k_2} = \frac{s}{l_1} \quad &\Longleftrightarrow \quad k_2 = \frac{2s}{s+l_1} > 1 \end{aligned}$$

Then in the country-1-default region, $k_2\sigma$ measure of agents invest in country 2, i.e. play $\phi = 0$, while $(2-k_2)\,\sigma$ measure of agents play $\phi = 1$. Similarly in the country-2-default region, , $k_1\sigma$ measure of agents play $\phi = 1$ while $(2-k_1)\,\sigma$ measure of agents play $\phi = 0$.

Conjecture the following equilibrium with cut off δ^*

$$\phi(y) = \begin{cases} .... \\ 1, & y \in [\delta^* - 2\sigma, \delta^* - k_2\sigma] \\ 0, & y \in [\delta^* - k_2\sigma, \delta^*] \\ 1, & y \in [\delta^*, \delta^* + k_1\sigma] \\ 0, & y \in [\delta^* + k_1\sigma, \delta^* + 2\sigma] \\ 1, & y \in [\delta^* + 2\sigma, \delta^* + 2\sigma + k_1\sigma] \\ .... \end{cases}$$

In other words, two types of equilibria collide at δ^*. I conjecture that marginal investor at δ^* is indifferent, while the agents between $[\delta^* - k_2\sigma, \delta^*]$ strictly prefer $\phi = 0$, and symmetrically the agents between $[\delta^*, \delta^* + k_1\sigma]$ strictly prefer $\phi = 1$. Other agents in this economy are indifferent.

Let x denote the fraction of agents with signal realization above the agent's private signal δ_j, so that given x, the true fundamental is

$$\delta(x) = \delta_j - (1-2x)\,\sigma$$

Further, let $\rho(\delta_j, x)$ be the expected proportion agents investing in country 1 given x. Then, we have

$$\rho(\delta_j, x) = \begin{cases} 1 - \frac{k_2}{2}, & \delta + 2\sigma x < \delta^* + (2-k_2)\,\sigma \\ x + cst, & else \\ \frac{k_1}{2} & \delta - 2\sigma(1-x) > \delta^* - (2-k_1)\,\sigma \end{cases}$$

where cst is picked so that $\rho(\delta_j, x)$ is continuous in x. We note that the slope is generically x as we are replacing

$\phi = 0$ with $\phi = 1$ marginally. At $\delta_j = \delta^*$, we have

$$\rho(\delta^*, x) = \begin{cases} 1 - \frac{k_2}{2}, & x < 1 - \frac{k_2}{2} \\ x, & else \\ \frac{k_1}{2} & x > \frac{k_1}{2} \end{cases}$$

and we need

$$1 - \frac{k_2}{2} < \frac{k_1}{2}$$

Note that if we assume that $\rho_{min}(\delta), 1 - \rho_{max}(\delta) \in \left[1 - \frac{k_2}{2}, \frac{k_1}{2}\right]$ we have a 1-to-1 function between x and ρ that yields

$$\begin{aligned} x_{min} = \frac{1 - \theta_1(\delta^*)}{1+f} = \frac{1-\theta}{1+f} e^{-\delta^*} \quad &\Longleftrightarrow \quad \ln x_{min} = -z - \delta^* \\ 1 - x_{max} = s\frac{1 - \theta_2(\delta^*)}{1+f} = s\frac{1-\theta}{1+f} e^{\delta^*} \quad &\Longleftrightarrow \quad \ln(1 - x_{max}) = \ln s - z + \delta^* \end{aligned}$$

Note here that we are ignoring fundamental uncertainty. Otherwise, we need to take account of the fact that in the mind of the agent,

$$\rho_{min}(\delta(x)) = e^{-z} e^{-\delta(x)} = e^{-z} e^{-[\delta_j - (1-2x)\sigma]}$$

is the minimum investment in country 1 needed for it to survive conditional on x. For everything else below, we assume that $\rho_{min}(\delta(x)) = \rho_{min}(\delta_j)$. Next, note that

$$x = \text{Fraction of people with signal above agent}$$

so that $x = 1$ is the most pessimistic agent, and $x = 0$ is the most optimistic. As $\rho(\delta, x)$ is increasing in x, we have

$$\begin{aligned} x < x_{min} \quad &\Longleftrightarrow \quad \text{Country 1 fails} \\ x > x_{min} \quad &\Longleftrightarrow \quad \text{Country 1 survives} \\ x < x_{max} \quad &\Longleftrightarrow \quad \text{Country 2 survives} \\ x > x_{max} \quad &\Longleftrightarrow \quad \text{Country 2 fails} \end{aligned}$$

Then, for the boundary agent, the expected return of investing in country 2 is given by

$$\begin{aligned} \Pi_2(\delta^*) &= Return_2(survival) + Return_2(default) \\ &= \int_0^{x_{max}} \frac{s}{(1+f)(1 - \rho(\delta^*, x))} dx + \int_{x_{max}}^1 \frac{l_2 s}{(1+f)(1 - \rho(\delta^*, x))} dx \\ &= \int_0^{1-\frac{k_2}{2}} \frac{s}{(1+f)\left(1 - \left(1 - \frac{k_2}{2}\right)\right)} dx + \int_{1-\frac{k_2}{2}}^{x_{max}} \frac{s}{(1+f)(1-x)} \\ &\quad + \int_{x_{max}}^{\frac{k_1}{2}} \frac{l_2 s}{(1+f)(1-x)} dx + \int_{\frac{k_1}{2}}^1 \frac{l_2 s}{(1+f)\left(1 - \frac{k_1}{2}\right)} dx \\ &= \left(1 - \frac{k_2}{2}\right) \frac{s}{(1+f)\frac{k_2}{2}} + \frac{s}{1+f}\left[\ln\left(\frac{k_2}{2}\right) - \ln(1 - x_{max})\right] \\ &\quad + \frac{l_2 s}{1+f}\left[\ln(1 - x_{max}) - \ln\left(1 - \frac{k_1}{2}\right)\right] + \left(1 - \frac{k_1}{2}\right) \frac{l_2 s}{(1+f)\left(1 - \frac{k_1}{2}\right)} \\ &= \frac{s}{(1+f)} \left\{ \left(\frac{1 - \frac{k_2}{2}}{\frac{k_2}{2}}\right) + \left[\ln\left(\frac{k_2}{2}\right) - \ln(1 - x_{max})\right] + l_2 + l_2\left[\ln(1 - x_{max}) - \ln\left(1 - \frac{k_1}{2}\right)\right] \right\} \end{aligned}$$

and the expected return of investing in country 1 is given by

$$
\begin{aligned}
\Pi_1(\delta^*) &= \int_0^{x_{min}} \frac{l_1}{(1+f)\,\rho(\delta^*,x)}dx + \int_{x_{min}}^1 \frac{1}{(1+f)\,\rho(\delta^*,x)}dx \\
&= \int_0^{1-\frac{k_2}{2}} \frac{l_1}{(1+f)\left(1-\frac{k_2}{2}\right)}dx + \int_{1-\frac{k_2}{2}}^{x_{min}} \frac{l_1}{(1+f)\,x}dx \\
&\quad + \int_{x_{min}}^{\frac{k_1}{2}} \frac{1}{(1+f)\,x}dx + \int_{\frac{k_1}{2}}^1 \frac{1}{(1+f)\,\frac{k_1}{2}}dx \\
&= \left(1-\frac{k_2}{2}\right)\frac{l_1}{(1+f)\left(1-\frac{k_2}{2}\right)} + \frac{l_1}{1+f}\left[\ln(x_{min}) - \ln\left(1-\frac{k_2}{2}\right)\right] \\
&\quad + \frac{1}{1+f}\left[\ln\left(\frac{k_1}{2}\right) - \ln(x_{min})\right] + \left(1-\frac{k_1}{2}\right)\frac{1}{(1+f)\,\frac{k_1}{2}} \\
&= \frac{1}{1+f}\left\{l_1 + l_1\left[\ln(x_{min}) - \ln\left(1-\frac{k_2}{2}\right)\right] + \left[\ln\left(\frac{k_1}{2}\right) - \ln(x_{min})\right] + \left(\frac{1-\frac{k_1}{2}}{\frac{k_1}{2}}\right)\right\}
\end{aligned}
$$

Note that

$$
\begin{aligned}
\left(\frac{1-\frac{k_1}{2}}{\frac{k_1}{2}}\right) = \left(\frac{1}{\frac{k_1}{2}} - 1\right) &= 1 + sl_2 - 1 = sl_2 \\
\left(\frac{1-\frac{k_2}{2}}{\frac{k_2}{2}}\right) = \left(\frac{1}{\frac{k_2}{2}} - 1\right) &= \frac{s+l_1}{s} - \frac{s}{s} = \frac{l_1}{s}
\end{aligned}
$$

Setting these equal, we have

$$
\begin{aligned}
& s\left\{\frac{l_1}{s} + \left[\ln\left(\frac{k_2}{2}\right) - \ln(1-x_{max})\right] + l_2 + l_2\left[\ln(1-x_{max}) - \ln\left(1-\frac{k_1}{2}\right)\right]\right\} \\
= & \left\{l_1 + l_1\left[\ln(x_{min}) - \ln\left(1-\frac{k_2}{2}\right)\right] + \left[\ln\left(\frac{k_1}{2}\right) - \ln(x_{min})\right] + sl_2\right\}
\end{aligned}
$$

Plugging in for k_1, k_2 and

$$
\begin{aligned}
\frac{k_1}{2} &= \frac{1}{1+l_2 s} \\
\frac{k_2}{2} &= \frac{s}{s+l_1} \\
1-\frac{k_1}{2} &= \frac{l_2 s}{1+l_2 s} \\
1-\frac{k_2}{2} &= \frac{l_1}{s+l_1} \\
\ln(x_{min}) &= -z-\delta^* \\
\ln(1-x_{max}) &= -z+\delta^*+\ln s
\end{aligned}
$$

Setting these equal, we have

$$
\begin{aligned}
& s\left\{\left[\ln\left(\frac{k_2}{2}\right)-\ln\left(1-x_{max}\right)\right]+l_2\left[\ln\left(1-x_{max}\right)-\ln\left(1-\frac{k_1}{2}\right)\right]\right\} \\
& \qquad = l_1\left[\ln\left(x_{min}\right)-\ln\left(1-\frac{k_2}{2}\right)\right]+\left[\ln\left(\frac{k_1}{2}\right)-\ln\left(x_{min}\right)\right] \\
\iff & s\left\{-\left(1-l_2\right)\ln\left(1-x_{max}\right)+\left[\ln\left(\frac{k_2}{2}\right)-l_2\ln\left(1-\frac{k_1}{2}\right)\right]\right\} \\
& \qquad = -\left(1-l_1\right)\ln\left(x_{min}\right)+\left[\ln\left(\frac{k_1}{2}\right)-l_1\ln\left(1-\frac{k_2}{2}\right)\right] \\
\iff & s\left\{\left(1-l_2\right)\left(z-\delta^*-\ln s\right)+\left[\ln\left(\frac{s}{s+l_1}\right)-l_2\ln\left(\frac{l_2 s}{1+l_2 s}\right)\right]\right\} \\
& \qquad = \left(1-l_1\right)\left(z+\delta^*\right)+\left[\ln\left(\frac{1}{1+l_2 s}\right)-l_1\ln\left(\frac{l_1}{s+l_1}\right)\right]
\end{aligned}
$$

Finally, solving for δ^*, we have

$$
\begin{aligned}
\delta^* &= \frac{s\left\{\left(1-l_2\right)\left(z-\ln s\right)+\left[\ln\left(\frac{s}{s+l_1}\right)-l_2\ln\left(\frac{l_2 s}{1+l_2 s}\right)\right]\right\}-\left(1-l_1\right)z-\left[\ln\left(\frac{1}{1+l_2 s}\right)-l_1\ln\left(\frac{l_1}{s+l_1}\right)\right]}{\left(1-l_1\right)+s\left(1-l_2\right)} \\
&= \frac{s\left\{\left(1-l_2\right)z-\left(1-l_2\right)\ln s+\ln s-\ln\left(s+l_1\right)-l_2\ln l_2-l_2\ln s+l_2\ln\left(1+l_2 s\right)\right\}}{\left(1-l_1\right)+s\left(1-l_2\right)} \\
&\quad +\frac{-\left(1-l_1\right)z+\ln\left(1+l_2 s\right)+l_1\ln\left(l_1\right)-l_1\ln\left(s+l_1\right)}{\left(1-l_1\right)+s\left(1-l_2\right)}
\end{aligned}
$$

so that finally

$$
\delta^* = \frac{\left[\left(1-l_2\right)s-\left(1-l_1\right)\right]z-\left(s+l_1\right)\ln\left(s+l_1\right)+\left(1+sl_2\right)\ln\left(1+l_2 s\right)+l_1\ln l_1-sl_2\ln l_2}{\left(1-l_1\right)+s\left(1-l_2\right)} \tag{B.2}
$$

Plugging in $l_1 = l_2 = 0$, we have

$$
\delta^* = \frac{-\left(1-s\right)z-s\ln\left(s\right)}{1+s}
$$

just as we had before.

We want to show that from the perspective of δ^*, for an x small enough so that $\rho\left(\delta^*, x\right) = 1-\frac{k_2}{2}$, does country 1 default? We know that $\rho_{min}\left(\delta^*\right) = e^{-z}e^{-\delta^*}$, so that

$$
\begin{aligned}
\rho_{min}\left(\delta^*\right) &> 1-\frac{k_2}{2} \\
\iff \ln\left(\rho_{min}\left(\delta^*\right)\right) &> \ln\left(1-\frac{k_2}{2}\right) \\
\iff -\left(\delta^*+z\right) &> \ln\left(\frac{l_1}{s+l_1}\right)
\end{aligned}
$$

which gives

$$
\begin{aligned}
& -\left[2\left(1-l_2\right)sz-\left(s+l_1\right)\ln\left(s+l_1\right)+\left(1+sl_2\right)\ln\left(1+l_2 s\right)+l_1\ln l_1-sl_2\ln l_2\right] \\
> \ & \left[\left(1-l_1\right)+s\left(1-l_2\right)\right]\left[\ln l_1-\ln\left(s+l_1\right)\right]
\end{aligned}
$$

and ultimately yields

$$
F_1^*\left(l_1, l_2, s\right)] \equiv -2\left(1-l_2\right)sz-\left[1+s\left(1-l_2\right)\right]\ln l_1+sl_2\ln l_2+\left[1+s\left(2-l_2\right)\right]\ln\left(s+l_1\right)-\left(1+l_2 s\right)\ln\left(1+l_2 s\right)
$$

and the default condition is given by $F_1^*(l_1, l_2, s) \geq 0$. Assume $l_1 = l_2 = l$. Then, we have

$$F_1^*(l, l, s) = -2(1-l)sz - [1-(1-2l)s]\ln l + [1+s(2-l)]\ln(s+l) - (1+ls)\ln(1+ls)$$

We can show that $F_1^*(l, l, s)$ is always positive for small enough recovery l as the term $-[1-(1-2l)s]\ln l$ explodes, swamping any negative z effect.[12]

Next, we want to show that from the perspective of δ^*, for an x large enough so that $\rho(\delta^*, x) = \frac{k_1}{2}$, does country 2 default? We know that $1 - \rho_{max}(\delta^*) = se^{-z}e^{\delta^*}$, so that

$$\begin{aligned} 1 - \rho_{max}(\delta^*) &> 1 - \frac{k_1}{2} \\ \iff \ln(1 - \rho_{max}(\delta^*)) &> \ln\left(1 - \frac{k_1}{2}\right) \\ \iff \ln s - z + \delta^* &> \ln\left(\frac{l_2 s}{1 + l_2 s}\right) \end{aligned}$$

so that

$$\begin{aligned} & [(1-l_1) + s(1-l_2)]\ln s - 2(1-l_1)z - (s+l_1)\ln(s+l_1) + (1+sl_2)\ln(1+l_2 s) + l_1 \ln l_1 - sl_2 \ln l_2 \\ > \ & [(1-l_1) + s(1-l_2)][\ln l_2 + \ln s - \ln(1+l_2 s)] \end{aligned}$$

Define

$$F_2^*(l_1, l_2, s) \equiv -2(1-l_1)z - (s+l_1)\ln(s+l_1) + (2-l_1+s)\ln(1+l_2 s) + l_1 \ln l_1 - [s+(1-l_1)]\ln l_2$$

and the default condition is given by $F_2^*(l_1, l_2, s) \geq 0$. Assuming equal recovery $l_1 = l_2 = l$, we have

$$F_2^*(l, l, s) = -2(1-l)z - (s+l)\ln(s+l) + (2-l+s)\ln(1+ls) - [s+(1-2l)]\ln l$$

We can show that $F_2^*(l, l, s)$ is always positive for small enough recovery l as the term $-[s+(1-2l)]\ln l$ explodes, swamping any negative z effect.

Let us consider an interior agent, i.e., $\delta \in [\delta^* - k_2\sigma, \delta^* + k_1\sigma]$. Let

$$\delta(\varepsilon) = \delta^* + 2\varepsilon\sigma$$

with $\varepsilon \in \left[-\frac{k_2}{2}, \frac{k_1}{2}\right]$. Let us first consider investment in country 1. We have $\rho_{min}(\delta)$ as the default boundary, and actual investment is given by

$$\rho(\delta, x) = \begin{cases} 1 - \frac{k_2}{2}, & \delta^* + \varepsilon 2\sigma + 2\sigma x < \delta^* + (2-k_2)\sigma \\ x + cst, & else \\ \frac{k_1}{2} & \delta^* + \varepsilon 2\sigma - 2\sigma(1-x) > \delta^* - (2-k_1)\sigma \end{cases} = \begin{cases} 1 - \frac{k_2}{2}, & 2\varepsilon\sigma + 2\sigma x < (2-k_2)\sigma \\ x + cst, & else \\ \frac{k_1}{2} & 2\varepsilon\sigma - 2\sigma(1-x) > -(2-k_1)\sigma \end{cases}$$

which gives

$$\rho(\delta, x) = \begin{cases} 1 - \frac{k_2}{2}, & \varepsilon + x < 1 - \frac{k_2}{2} \\ x + \varepsilon, & else \\ \frac{k_1}{2} & \varepsilon + x > \frac{k_1}{2} \end{cases}$$

[12] Taking derivatives w.r.t. l and s, we have

$$\begin{aligned} \partial_l F_1^*(l, l, s) &= 2sz + s - \frac{(1+s)}{l} + \frac{1+(2-l)s}{s+l} + 2s\ln l - s\ln(s+l) - s\ln(1+ls) \\ \partial_s F_1^*(l, l, s) &= \frac{1+(2-l)s}{s+l} - l\ln(1+ls) + (2-l)\ln(s+l) - 2(1-l)z - l - (1-2l)\ln l \end{aligned}$$

Note that we have $cst = \varepsilon$ by imposing continuity (which has to follow from $\rho(\delta, x)$ being an integral over strategies ϕ).

Let $x_{min}(\delta)$ be the lowest $x \in [0,1]$ such that

$$\rho(\delta, x) = \varepsilon + x \geq \rho_{min}(\delta)$$

and we therefore have

$$x_{min}(\delta) = \max\{\rho_{min}(\delta) - \varepsilon, 0\}$$

Similarly, let $x_{max}(\delta)$ be the highest $x \in [0,1]$ such that

$$1 - \rho(\delta, x) = 1 - \varepsilon - x \geq 1 - \rho_{max}(\delta)$$

and thus

$$1 - x_{max}(\delta) = \max\{1 - \rho_{max}(\delta) + \varepsilon, 0\}$$

The expected return of investing in country 1 is then given by

$$\begin{aligned}
\Pi_1(\delta) &= \int_{x:\rho(\delta,x)<\rho_{min}(x)} \frac{l_1}{(1+f)\rho(\delta,x)}dx + \int_{x:\rho(\delta,x)\geq\rho_{min}(x)} \frac{1}{(1+f)\rho(\delta,x)}dx \\
&= \int_0^{x_{min}(\delta)} \frac{l_1}{(1+f)\rho(\delta,x)}dx + \int_{x_{min}(\delta)}^1 \frac{1}{(1+f)\rho(\delta,x)}dx \\
&= \int_0^{1-\frac{k_2}{2}-\varepsilon} \frac{l_1}{(1+f)\left(1-\frac{k_2}{2}\right)}dx + \int_{1-\frac{k_2}{2}-\varepsilon}^{x_{min}(\delta)} \frac{l_1}{(1+f)(x+\varepsilon)}dx \\
&\quad + \int_{x_{min}(\delta)}^{\frac{k_1}{2}-\varepsilon} \frac{1}{(1+f)(x+\varepsilon)}dx + \int_{\frac{k_1}{2}-\varepsilon}^1 \frac{1}{(1+f)\frac{k_1}{2}}dx \\
&= \frac{l_1}{1+f}\left[\frac{1-\frac{k_2}{2}-\varepsilon}{1-\frac{k_2}{2}} + \ln(x_{min}(\delta)+\varepsilon) - \ln\left(1-\frac{k_2}{2}\right)\right] \\
&\quad + \frac{1}{1+f}\left[\ln\left(\frac{k_1}{2}\right) - \ln(x_{min}(\delta)+\varepsilon) + \frac{1-\frac{k_1}{2}+\varepsilon}{\frac{k_1}{2}}\right] \\
&= \frac{l_1}{1+f}\left[1 - \frac{\varepsilon}{1-\frac{k_2}{2}} + \ln(x_{min}(\delta)+\varepsilon) - \ln\left(1-\frac{k_2}{2}\right)\right] \\
&\quad + \frac{1}{1+f}\left[\ln\left(\frac{k_1}{2}\right) - \ln(x_{min}(\delta)+\varepsilon) + \frac{1-\frac{k_1}{2}}{\frac{k_1}{2}} + \frac{\varepsilon}{\frac{k_1}{2}}\right] \\
&= \Pi_1(\delta^*) + \frac{l_1}{1+f}\left[-\frac{\varepsilon}{1-\frac{k_2}{2}} + \ln(x_{min}(\delta)+\varepsilon) - \ln x_{min}(\delta^*)\right] \\
&\quad + \frac{1}{1+f}\left[\ln x_{min}(\delta^*) - \ln(x_{min}(\delta)+\varepsilon) + \frac{\varepsilon}{\frac{k_1}{2}}\right] \\
&= \Pi_1(\delta^*) + \frac{1}{1+f}\left\{\varepsilon\left(\frac{1}{\frac{k_1}{2}} - \frac{l_1}{1-\frac{k_2}{2}}\right) - (1-l_1)\left[\ln(x_{min}(\delta)+\varepsilon) - \ln x_{min}(\delta^*)\right]\right\} \\
&= \Pi_1(\delta^*) + \frac{1}{1+f}\left\{\varepsilon\left[(1-l_1) - s(1-l_2)\right] - (1-l_1)\left[\ln(x_{min}(\delta)+\varepsilon) - \ln x_{min}(\delta^*)\right]\right\}
\end{aligned}$$

Similarly, investing in country 2 gives

$$
\begin{aligned}
\Pi_2(\delta) &= \int_0^{x_{max}(\delta)} \frac{s}{(1+f)(1-\rho(\delta,x))}dx + \int_{x_{max}(\delta)}^1 \frac{l_2 s}{(1+f)(1-\rho(\delta,x))}dx \\
&= \int_0^{1-\frac{k_2}{2}-\varepsilon} \frac{s}{(1+f)\left(1-\left(1-\frac{k_2}{2}\right)\right)}dx + \int_{1-\frac{k_2}{2}-\varepsilon}^{x_{max}(\delta)} \frac{s}{(1+f)(1-x-\varepsilon)} \\
&\quad + \int_{x_{max}(\delta)}^{\frac{k_1}{2}-\varepsilon} \frac{l_2 s}{(1+f)(1-x-\varepsilon)}dx + \int_{\frac{k_1}{2}-\varepsilon}^1 \frac{l_2 s}{(1+f)\left(1-\frac{k_1}{2}\right)}dx \\
&= \frac{s}{1+f}\left[\frac{1-\frac{k_2}{2}-\varepsilon}{\frac{k_2}{2}} + \ln\left(\frac{k_2}{2}\right) - \ln(1-x_{max}(\delta)-\varepsilon)\right] \\
&\quad + \frac{sl_2}{1+f}\left[\ln(1-x_{max}(\delta)-\varepsilon) - \ln\left(1-\frac{k_1}{2}\right) + \frac{1-\frac{k_1}{2}+\varepsilon}{1-\frac{k_1}{2}}\right] \\
&= \Pi_2(\delta^*) + \frac{s}{1+f}\left\{\varepsilon\left(l_2\frac{1}{1-\frac{k_1}{2}} - \frac{1}{\frac{k_2}{2}}\right) + (1-l_2)\left[\ln(1-x_{max}(\delta^*)) - \ln(1-x_{max}(\delta)-\varepsilon)\right]\right\} \\
&= \Pi_2(\delta^*) + \frac{s}{1+f}\left\{\varepsilon\left[\frac{(1-l_1)-s(1-l_2)}{s}\right] + (1-l_2)\left[\ln(1-x_{max}(\delta^*)) - \ln(1-x_{max}(\delta)-\varepsilon)\right]\right\}
\end{aligned}
$$

Let us define

$$
\begin{aligned}
g(\varepsilon) &\equiv (1+f)\left[\Pi_1(\delta) - \Pi_2(\delta)\right] \\
&= \varepsilon\left[(1-l_1)-s(1-l_2)\right] - (1-l_1)\left[\ln(x_{min}(\delta)+\varepsilon) - \ln x_{min}(\delta^*)\right] \\
&\quad -s\left\{\varepsilon\left[\frac{(1-l_1)-s(1-l_2)}{s}\right] + (1-l_2)\left[\ln(1-x_{max}(\delta^*)) - \ln(1-x_{max}(\delta)-\varepsilon)\right]\right\} \\
&= -(1-l_1)\left[\ln(x_{min}(\delta^*+2\sigma\varepsilon)+\varepsilon) - \ln x_{min}(\delta^*)\right] \\
&\quad +s(1-l_2)\left[\ln(1-x_{max}(\delta^*+2\sigma\varepsilon)-\varepsilon) - \ln(1-x_{max}(\delta^*))\right] \\
&\quad +\varepsilon\left\{\left[(1-l_1)-s(1-l_2)\right] - s\left[\frac{(1-l_1)-s(1-l_2)}{s}\right]\right\} \\
&= -(1-l_1)\left[\ln(x_{min}(\delta^*+2\sigma\varepsilon)+\varepsilon) - \ln x_{min}(\delta^*)\right] \\
&\quad +s(1-l_2)\left[\ln(1-x_{max}(\delta^*+2\sigma\varepsilon)-\varepsilon) - \ln(1-x_{max}(\delta^*))\right]
\end{aligned}
$$

Taking the derivative w.r.t. ε, we have many different cases. The issue is if x_{min} or x_{max} start binding first. Regardless, close to $\varepsilon = 0$ we have neither x_{min} or x_{max} cornered, so that

$$
\begin{aligned}
\ln(x_{min}(\delta^*+2\sigma\varepsilon)+\varepsilon) &= \ln(\rho_{min}(\delta(\varepsilon))) = -z-\delta(\varepsilon) = -z-(\delta^*+2\sigma\varepsilon) \\
\ln(1-x_{max}(\delta^*+2\sigma\varepsilon)-\varepsilon) &= \ln(1-\rho_{max}(\delta(\varepsilon))) = s\ln s - z + \delta(\varepsilon) = s\ln s - z + (\delta^*+2\sigma\varepsilon)
\end{aligned}
$$

and thus for ε small we have

$$
g'(\varepsilon) = -(1-l_1)(-)2\sigma + s(1-l_2)2\sigma = 2\sigma\left[(1-l_1)+s(1-l_2)\right] > 0
$$

and indeed we have the incentives of the agents aligned with the conjectured strategies, at least around δ^*.

Next, we have to account for all the different cases – that is, we know that at some distance ε that x_{min}, x_{max} start binding at $0, 1$, respectively.

Let ε_{min} be the point at which x_{min} becomes cornered, that is

$$
\rho_{min}(\delta) = \varepsilon \iff e^{-z}e^{-(\delta^*+2\sigma\varepsilon)} = \varepsilon \iff 2\sigma\varepsilon + \ln\varepsilon = -z-\delta^*
$$

Note that $\rho_{min}(\delta) > 0$ so that there is no solution for $\varepsilon < 0$.

Similarly, let ε_{max} be the point at which x_{max} becomes cornered, that is

$$1-\rho_{max}(\delta)=-\varepsilon \iff se^{-z}e^{\delta^*+2\sigma\varepsilon}=-\varepsilon \iff 2\sigma(-\varepsilon)+\ln(-\varepsilon)=\ln s-z+\delta^*$$

Note that $1-\rho_{max}(\delta)\geq 0$ so that there is no solution for $\varepsilon>0$.

Positive ε. Consider positive ε. Thus, we only have to worry about x_{min} cornered. When x_{min} becomes cornered, then

$$\frac{\partial}{\partial\varepsilon}\ln\left(x_{min}\left(\delta^*+2\sigma\varepsilon\right)+\varepsilon\right)=\frac{1}{\varepsilon}$$

Then, we have

$$g'(\varepsilon)=-(1-l_1)\frac{1}{\varepsilon}+s(1-l_2)2\sigma$$

The derivative is increasing in ε, and is largest at $\varepsilon=\frac{k_1}{2}$ at a value of

$$g'\left(\frac{k_1}{2}\right)=-(1-l_1)(1+l_2s)+s(1-l_2)2\sigma$$

For small enough σ, this is always negative.

Negative ε. Consider negative ε. Thus, we only have to worry about x_{max} cornered. When x_{max} becomes cornered, then

$$\frac{\partial}{\partial\varepsilon}\ln\left(1-x_{max}\left(\delta^*+2\sigma\varepsilon\right)-\varepsilon\right)=-\frac{1}{\varepsilon}$$

Then, we have

$$g'(\varepsilon)=(1-l_1)2\sigma+s(1-l_2)\left(-\frac{1}{\varepsilon}\right)$$

The derivative is again increasing in ε, and is largest at $\varepsilon=-\frac{k_2}{2}$ at a value of

$$g'\left(-\frac{k_2}{2}\right)=-(1-l_2)(s+l_1)+(1-l_1)2\sigma$$

For small enough σ, this is always negative.

For $s=1$ and $l_1=l_2=l$, we have symmetric conditions.

The last thing we need to do is to check that

$$g\left(-\frac{k_2}{2}\right)=g(0)=g\left(\frac{k_1}{2}\right)=0$$

To this end, we can also proof that as $\sigma\to 0$, indeed one country (which one depending on on which side of δ^* the realization of δ falls) will always default. This is equivalent to the interior assumption for x_{max}, x_{min} we made. For this to hold, we need the following restrictions

$$1-\frac{k_1}{2}\leq 1-\rho_{max}(\delta^*)\leq\frac{k_2}{2} \tag{B.3}$$

$$1-\frac{k_2}{2}\leq \rho_{min}(\delta^*)\leq\frac{k_1}{2} \tag{B.4}$$

The first line says that as $\sigma\to 0$, if $\delta<\delta^*$ then a proportion $\frac{k_2}{2}$ of investors invests in country 2, and it survives. However, if $\delta>\delta^*$, then only a proportion $1-\frac{k_1}{2}$ of investors invests in country 2, and it defaults. Similar arguments hold for country 1, which is summarized by the second line.

This can be rewritten as

$$\begin{aligned}
\ln\left(1-\frac{k_1}{2}\right) &\leq \ln\left(1-\rho_{max}\left(\delta^*\right)\right) \leq & \ln\left(\frac{k_2}{2}\right) \\
\ln\left(1-\frac{k_2}{2}\right) &\leq \ln\rho_{min}\left(\delta^*\right) \leq & \ln\left(\frac{k_1}{2}\right)
\end{aligned}$$

which gives

$$\begin{aligned}
\ln\left(\frac{l_2 s}{1+l_2 s}\right) &\leq \ln s - z + \delta^* \leq & \ln\left(\frac{s}{s+l_1}\right) \\
\ln\left(\frac{l_1}{s+l_1}\right) &\leq -z - \delta^* \leq & \ln\left(\frac{1}{1+l_2 s}\right)
\end{aligned}$$

equivalent to

$$\begin{aligned}
\ln\left(\frac{l_2}{1+l_2 s}\right)+z &\leq \delta^* \leq & \ln\left(\frac{1}{s+l_1}\right)+z \\
\ln\left(\frac{l_1}{s+l_1}\right)+z &\leq -\delta^* \leq & \ln\left(\frac{1}{1+l_2 s}\right)+z
\end{aligned}$$

equivalent to

$$\begin{aligned}
\ln\left(l_2\right)-\ln\left(1+l_2 s\right)+z &\leq \delta^* \leq & -\ln\left(s+l_1\right)+z \\
-\ln\left(\frac{1}{1+l_2 s}\right)-z &\leq \delta^* \leq & -\ln\left(\frac{l_1}{s+l_1}\right)-z
\end{aligned}$$

equivalent to

$$\begin{aligned}
\ln\left(l_2\right)-\ln\left(1+l_2 s\right)+z &\leq \delta^* \leq & -\ln\left(s+l_1\right)+z \\
\ln\left(1+l_2 s\right)-z &\leq \delta^* \leq & \ln\left(s+l_1\right)-\ln\left(l_1\right)-z
\end{aligned}$$

so that finally

$$\max\left[\ln\left(l_2\right)-\ln\left(1+l_2 s\right)+z, \ln\left(1+l_2 s\right)-z\right] \leq \delta^* \leq \min\left[-\ln\left(s+l_1\right)+z, \ln\left(s+l_1\right)-\ln\left(l_1\right)-z\right] \tag{B.5}$$

The first term is binding on the RHS for $z > \ln\left(1+l_2 s\right) - \frac{1}{2}\ln\left(l_2\right)$, and the first term is binding on the left hand side for $z < \ln\left(s+l_1\right) - \frac{1}{2}\ln\left(l_1\right)$.