UNIL | Université de Lausanne
Département de finance

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

FINANCE RESEARCH SEMINAR SUPPORTED BY UNIGESTION

Dimensional Analysis, Leverage Neutrality, and Market Microstructure Invariance

Anna OBIZHAEVA,
New Economic School, Moscow

Friday, October 11, 2019, 10:30-12:00
Room 126, Extranef building at the University of Lausanne

UNIGESTION

Dimensional Analysis, Leverage Neutrality, and Market Microstructure Invariance

Albert S. Kyle and Anna A. Obizhaeva*

First Draft: September 17, 2015

This Draft: July 30, 2017

This paper combines dimensional analysis, leverage neutrality, and a principle of market microstructure invariance to derive scaling laws expressing transaction costs functions, bid-ask spreads, bet sizes, number of bets, and other financial variables in terms of dollar trading volume and volatility. The scaling laws are illustrated using data on bid-ask spreads and number of trades for Russian and U.S. stocks. These scaling laws provide practical metrics for risk managers and traders; scientific benchmarks for evaluating controversial issues related to high frequency trading, market crashes, and liquidity measurement; and guidelines for designing policies in the aftermath of financial crisis.

JEL: G10, G12, G14, G20

Keywords: market microstructure, liquidity, bid-ask spread, trade size, market depth, dimensional analysis, leverage, invariance, econophysics

* Kyle: University of Maryland, College Park, MD 20742, USA, akyle@rhsmith.umd.edu. Obizhaeva: New Economic School, Moscow, Skolkovo, 143026, Russia, aobizhaeva@nes.ru. The authors thank Bo Hu and Thomas James for helpful comments as well as Sylvain Delalay for his assistance.

This paper combines dimensional analysis, leverage neutrality, and a market microstructure invariance hypothesis to derive scaling laws for the specification of transaction cost functions, the width of bid-ask spreads, the size distribution of bets or trades, the speed of bet or trade execution, the size of margin requirements and haircuts, optimal minimum increments of price fluctuations (tick size), and optimal increments of traded quantities (minimum lot size). The basic liquidity measure is proportional to the cube root of the ratio of dollar volume to return variance. Bid-ask spreads, which measure the difference between the highest price at which a trader is willing to buy ("bid") and the lowest price at which a trader is willing to sell ("offer"), are predicted to be inversely proportional to this liquidity measure. The rate at which trades arrive is predicted to be proportional to the product of the liquidity measure squared and volatility squared. The scaling is illustrated by showing that both bid-ask spreads and the number of trades indeed scale as predicted in both the Russian and U.S. stock markets.

In financial markets, institutional investors trade by implementing speculative "bets" which move prices. A bet is a decision to buy or sell a quantity of institutional size. In the stock market, traders execute bets by dividing them into separate orders, shredding the orders into small pieces, and executing numerous smaller quantities over time. Across different stocks, the time frame of execution may be minutes, hours, days, or weeks. The fields of market microstructure and econophysics use different but complementary approaches for studying how prices result from trading securities. As emphasized by Gabaix et al. (2003), large trades incur trading costs by moving prices. Foucault, Pagano and Roell (2013) and Bouchaud, Farmer and Lillo (2009) summarize various findings from the perspectives of market microstructure and econophysics, respectively.

Dimensional analysis simplifies scientific inference by imposing restrictions on unknown and potentially complicated relationships among different variables. In physics, researchers obtain powerful results by using dimensional analysis to reduce the dimensionality of problems, as reviewed in Barenblatt (1996). For example, Kolmogorov (1941) proposed a simple dimensional analysis argument to derive his "5/3–law" for the energy distribution in a turbulent fluid. Dimensional analysis can be also used to infer the size and number of molecules in a mole of gas or the size of the explosive energy in an atomic blast from measurable large-scale physical quantities. In this paper, we apply dimensional analysis to an economic problem using an approach which mimics the way physicists apply dimensional analysis to physics problems. This paper derives new results by using dimensional analysis to relate market liquidity to dollar volume and returns volatility. Our analysis takes place in three steps which mimic the use of dimensional analysis in physics.

First, in physics, dimensional analysis begins with fundamental units of mass, distance, and time. In finance, dimensional analysis begins with fundamental units of time, value, and asset quantity. The problem is simplified by constructing dimensionless ratios of dimensional quantities measuring asset prices, trading volumes, returns volatilities, and costs of making bets.

Second, physics researchers augment dimensional analysis with conservation laws based on principles of physics, such as the law of conservation of energy. In finance, proceeding further requires introducing conservation laws based on principles of finance. Since no-arbitrage principles—like Black and Scholes (1973) use to derive their option pricing model—are so fundamental to finance, the financial conservation laws naturally take the form of no-arbitrage restrictions. Here we use the less restrictive, more simplistic no-arbitrage principle of leverage neutrality. Since cash is a risk-free asset, transfers of cash between traders occur at zero cost; riskless, cash-equivalent assets are infinitely liquid. If a risky asset is combined with a positive or negative amount of a cash-equivalent asset and this bundle then is traded as a single package, the economics behind trading this package does not depend on how much cash is included into it. This does not create arbitrage opportunities.

Leverage neutrality captures the intuition of Modigliani and Miller (1958) that a firm's mix of equity and riskfree debt securities does not affect the value of a firm. Leverage neutrality implies that changes in a security's volatility, resulting from the amount of riskfree debt used to finance a firm, does not affect the economic outcomes associated with execution of bets that transfer risks embedded in the firm's securities. Leverage neutrality is also related to the idea that changes in margin requirements and repo haircuts do not affect the costs of transferring risks.

Third, physics researchers often take as given dimensional constants which do not vary during the analysis, such as the acceleration of gravity at sea level on the planet earth. Here we make the invariance assumption that the expected dollar cost of executing a bet is constant across assets and time. We refer to this assumption as market microstructure invariance. This leads to specific testable empirical scaling hypotheses with exponents of 1/3 or 2/3. In market microstructure, dimensional analysis leads to new insights which are neither obvious nor well-known. For example, predictions about microscopic financial quantities like the size of bets and the width of bid-ask spreads can be made based on observing macroscopic quantities like aggregate dollar volume and returns volatility.

As an alternative to dimensional analysis, analogous results can be derived in two different ways: (1) directly from empirical invariance hypotheses or (2) as properties of an equilibrium model. While all three approaches are consistent with one another, each of them generates new insights about invariance and has some advantages over the other two.

First, Kyle and Obizhaeva (2016*c*) derive and test scaling laws using two empirical invariance conjectures based on analyzing trading in business time. They hypothesize that (1) the dollar risk transferred by bets and (2) the dollar costs of executing bets, transferring economically equivalent risks, are the same across securities and time. This approach implicitly incorporates leverage neutrality. Dimensional analysis is more general because it can easily be generalized to include other explanatory variables and to generate new quantitative predictions about variables of interest.

Second, Kyle and Obizhaeva (2016*b*) take the alternative approach of deriving

scaling laws as endogenous implications of a dynamic equilibrium model with a specific structure, which assumes that the effort required to generate one discrete bet does not vary across securities and time. This assumption is closely related to this paper's assumption of an invariant expected execution cost for each bet. Risk-neutral informed traders make bets whose frequency and size depend on expected trading profits constrained by endogenous market depth, not exogenous risk aversion. For analytical tractability, the model makes the restrictive assumption that traders make only one bet whose size is observed by liquidity providers taking the other side of the trade. Since dimensional analysis eschews making connections to specific micro-founded economic models, it suggests that our predictions may hold under more general assumptions.

When market observations deviate from what is expected, predictions derived from dimensional analysis provide a benchmark from which to interpret the economic meaning of the deviations. For example, a physicist might use a pendulum swinging in a vacuum as a benchmark from which to model the effects of the earth's atmosphere on a swinging pendulum. In market microstructure, deviations from benchmark predictions might result from omitting economically important variables or market frictions from the simplest specifications based on dimensional analysis. Dimensional analysis can therefore help evaluate the effects of omitted variables, such as time horizon of bet execution, or omitted frictions, such as minimum tick size and minimum lots size. It leads to a natural scientific process both for discovering new relationships among financial variables and for extending them in an internally consistent manner to include other explanatory variables. Using a generalized dimensional analysis algorithm, we augment a basic transaction cost formula by adding differences in execution horizons and variations in market frictions related to minimum tick size and minimum lot size.

Dimensional analysis per se does not guarantee success; rather, it helps to narrow alternatives to more promising paths for research. Leverage neutrality and microstructure invariance suggest additional restrictions based on the economics behind underlying processes. The ultimate check for validity of any theory is whether its predictions can be backed by empirical evidence. Our empirical evidence suggests that dimensional analysis does indeed establish good empirical benchmarks for analyzing trade size and bid-ask spreads.

Our predictions can be stated as power laws, which define log-linear relationships between finance quantities with specific exponents of 1/3 and 2/3; these relationships resemble familiar laws of physics. A related but different literature in finance and economics studies power laws which show up mostly in the tails of frequency distributions of financial variables such as trade size or returns. This literature is reviewed in Sornette (2004), Newman (2005), Gabaix et al. (2006), and Gabaix (2009). Gabaix et al. (2006) find an observed power law for trading volume and trade size of 3/2 and an observed power law of 3 for returns; they show that this is consistent with a square root model of price impact. The square root model of price impact is also consistent with the most parsimonious application of dimen-

sional analysis. An interesting issue for future research is to understand whether the specific exponents documented in this literature on power laws are consistent with the exponents of 1/3 and 2/3 that we derive here.

This paper is organized as follows. Section I discusses application of dimensional analysis to transaction cost modeling. Section II introduces leverage neutrality. Section III discusses the economic interpretation of some variables. Section IV describes market microstructure invariance. Section V presents empirical evidence using data from the Russian and U.S. stock markets. Section VI outlines a general algorithm for handling misspecified models with too many or too few variables. Section VII extends the analysis to other variables and concludes.

I. Dimensional Analysis.

In financial markets, traders exchange risky assets. Asset prices reveal information that influences resource allocation. Speculative traders develop trading ideas based on private information they acquire conducting securities research. Hedgers buy or sell risky assets to better allocate risk. Bets are sequences of speculative or hedging transactions to trade in the same direction based on the same—approximately independently distributed—information or motives.

Exchanging risks via trading securities is costly because execution of bets moves market prices. Buy bets push prices up and sell bets push prices down relative to pre-trade price benchmarks. This market impact occurs as a result of adverse selection. Since traders on the opposite sides of bets believe that bets may contain private information, they require a price concession as compensation. Transaction cost models quantify trading costs. Good transaction cost models are of great interest to traders.

We next use dimensional analysis to derive an internally consistent model of transaction costs. We follow the step-by-step procedure outlined by Barenblatt (1996) for non-finance applications. Dimensional analysis leads to scaling laws proportional to products of powers of explanatory variables with different exponents. Dimensional analysis pays careful attention to maintaining consistency of dimensions and units of measurement. In physics, the base dimensions are considered to be length, measured in meters; mass, measured in grams; and time, measured in seconds.

When finance researchers describe trading in financial markets, the base dimensions are *value*, measured in units of currency; *asset quantity*, measured in units of shares or contracts; and *time*, measured in units of years, months, days, hours, minutes, seconds, milliseconds, or even microseconds. In this paper, we measure value in U.S. dollars (USD) or Russian rubles (RUB), asset quantity in shares, and time in days.

Let P_{jt} denote the stock price, V_{jt} its share volume, and σ^2_{jt} its return variance, where the subscript jt refers to stock j at time t. Let Q_{jt} denote the number of shares traded in a bet, with $Q_{jt} > 0$ representing buying and $Q_{jt} < 0$ representing

selling. Let G_{jt} denote the expected price impact cost of executing the bet of Q_{jt} shares as a fraction of the unsigned value traded $P_{jt}\,|Q_{jt}|$, with $G_{jt} \geq 0$.

We next show how to apply dimensional analysis for transaction cost modeling to make predictions about G_{jt}. Applying dimensional analysis correctly requires selecting the right set of variables to construct a model which explains the variable of interest.

ASSUMPTION 1 (Dimensional Analysis): *The market impact cost G_{jt} of executing a bet of Q_{jt} shares is a function of only five variables: the number of shares Q_{jt}, the stock price P_{jt}, share volume V_{jt}, return variance σ^2_{jt}, and expected dollar "bet cost" C:*

$$G_{jt} := g(Q_{jt}, P_{jt}, V_{jt}, \sigma^2_{jt}, C). \tag{1}$$

Writing g instead of g_{jt} reflects the important assumption that the price impact cost is a function only of its parameters and not other characteristics of asset j at time t. The variable C is key in our invariance framework. It is defined as the unconditional expected dollar costs of executing a bet,

$$C := \mathrm{E}\{G_{jt}\, P_{jt}\, |Q_{jt}|\}. \tag{2}$$

The value of C may be difficult to observe empirically. The hypothesis of market microstructure invariance, discussed below, implies that C may be written without subscripts jt.[1] For now, we assume that equation (1) is a correctly specified model that does not omit important explanatory variables. Later, we illustrate conceptually how to improve a misspecified model by including omitted variables such as tick size or minimum lot size.

Dimensional analysis requires paying careful attention to consistency of the units in which these quantities are measured. Let brackets $[X]$ define an operator which gives the dimensions of a variable X. Using this notation, the function arguments Q_{jt}, P_{jt}, V_{jt}, σ^2_{jt}, and C are measured using units of currency, shares, and time as follows:

$$\begin{aligned} [G_{jt}] &= 1, & [V_{jt}] &= \text{shares/day}, \\ [Q_{jt}] &= \text{shares}, & [\sigma^2_{jt}] &= 1/\text{day}, \\ [P_{jt}] &= \text{currency/shares}, & [C] &= \text{currency}. \end{aligned}$$

The notation $[G_{jt}] = 1$ means that the quantity G_{jt} is dimensionless. If G_{jt} were measured in basis points[2] instead of a fraction of the value traded, we would write $[G_{jt}] = 10^{-4}$. The implications of dimensional analysis depend critically on the fact that daily return variance σ^2_{jt} is measured in units of days, and therefore daily

[1] In Kyle and Obizhaeva (2016*c*), the average dollar bet cost C is denoted $\bar{C}_B$, and the percentage cost G_{jt} of executing a bet of size Q is denoted $C_{jt}(Q)$.

[2] We adopt the usual convention that "percentages" are dimensionless quantities expressed as multiples of 10^{-2}, and "basis points" are dimensionless quantities expressed as multiples of 10^{-4}.

return standard deviation σ_{jt} is measured in units equal to the square root of a day:

$$[\sigma_{jt}] = 1/\text{day}^{1/2} \tag{3}$$

Here we essentially assume that the price process is described by a Lévy process with exponent 2.

To illustrate reasonable economic magnitudes, market impact cost G_{jt} might be 10 basis points, bet size Q_{jt} might be 2500 shares, price P_{jt} might be \$40 per share, volume V_{jt} might be one million shares per day, daily return variance σ_{jt}^2 might be 0.0004 per day with daily volatility σ_{jt} then 0.02 per square-root-of-a-day, and dollar bet cost C might be \$2000.

Since there are only three distinct dimensions—value, quantity, and time—and five dimensional quantities—Q_{jt}, P_{jt}, V_{jt}, σ_{jt}^2, and C—it is possible to form two independent dimensionless quantities that can be used to redefine the arguments of the function g in an equivalent manner, separating dimensional quantities from dimensionless ones. Without loss of generality, let L_{jt} and Z_{jt} denote these two dimensionless quantities, defined by

$$L_{jt} := \left(\frac{m^2\, P_{jt}\, V_{jt}}{\sigma_{jt}^2\, C}\right)^{1/3}, \qquad Z_{jt} := \frac{P_{jt}\, Q_{jt}}{L_{jt}\, C}. \tag{4}$$

Here m^2 is a dimensionless scaling constant. Writing m without subscript jt is justified by the invariance assumptions made below in section IV. The exponent of 1/3 in the definition of L_{jt} is chosen strategically for important reasons related to leverage neutrality, as discussed below (equation (7)).

Since L_{jt} and Z_{jt} are dimensionless, we write

$$[L_{jt}] = 1, \qquad [Z_{jt}] = 1.$$

Without loss of generality, we re-define the arguments of the function g so that it is written as $g(P_{jt}, Q_{jt}, \sigma_{jt}^2, L_{jt}, Z_{jt})$.

There is some freedom in re-defining arguments, but several properties need to be satisfied. The three arguments P_{jt}, Q_{jt}, and σ_{jt}^2 are dimensional quantities which trivially span the three dimensions of value, quantity, and time since Q_{jt} has units of shares, $P_{jt}Q_{jt}$ has units of currency, and $1/\sigma_{jt}^2$ has units of days. These three variables are dimensionally independent in the sense that none of them has a dimension that can be expressed in terms of the dimensions of the others. They are also complete in the sense that the dimensions of the remaining two variables V_{jt} and C can be expressed in terms of the dimensions of these three variables. The two arguments L_{jt} and Z_{jt} are independent dimensionless quantities in the sense that V_{jt} and C can be recovered as functions of P_{jt}, Q_{jt}, σ_{jt}^2, L_{jt}, and Z_{jt}.

Since the value of $g(P_{jt}, Q_{jt}, \sigma_{jt}^2, L_{jt}, Z_{jt})$ is itself dimensionless, consistency of units implies that it cannot depend on the dimensional quantities P_{jt}, Q_{jt}, and σ_{jt}^2. Thus, *dimensional analysis* implies that the function g can be further simplified by

writing it as $g(L_{jt}, Z_{jt})$. The intuition here is that a physical law is independent of the units used to measure variables. The Buckingham π-theorem provides a formal justification for this approach.

PROPOSITION 1: *If the market impact function $G_{jt} := g(\ldots)$ is correctly specified as a function of the number of shares Q_{jt}, the stock price P_{jt}, share volume V_{jt}, the return variance σ^2_{jt}, and the bet cost C, then dimensional analysis implies that this function of five variables can be expressed as a function of two dimensional quantities by writing*

$$G_{jt} := g(Q_{jt}, P_{jt}, V_{jt}, \sigma^2_{jt}, C) = g(L_{jt}, Z_{jt}), \tag{5}$$

where the dimensionless variables L_{jt} and Z_{jt} are defined in equation (4).

This implication of dimensional analysis is based on the simple assumption that investors are not confused by units of measurement. In the context of a rational model, this implies that investors do not suffer money illusion, do not change their behavior when shares are split, and do not confuse calendar time with business time. For example, measuring the stock price in euros, rubles, or pennies generates the same transaction cost as measuring exactly the same stock price in dollars. To the extent that research in behavioral finance questions rationality, dimensional analysis provides the appropriate rational benchmark against which predictions of behavioral finance may be measured.

II. Leverage Neutrality.

To refine the transaction cost model further, we introduce a conservation law in the form of *leverage neutrality*. We assume existence of a cash-equivalent asset and rely on one of the fundamental properties of cash: Since a cash-equivalent asset embeds no risk, it can be exchanged in the market at no cost. This property of cash makes it suitable for being both a medium of exchange and a store of value.

ASSUMPTION 2 (Leverage Neutrality I): *Exchanging cash-equivalent assets incurs zero cost. Exchanging risky securities is costly. The economic cost of trading bundles of risky securities and cash-equivalent assets is the same for any positive or negative amount of cash-equivalent assets included into a bundle.*

Suppose that cash worth $P_{jt}(A-1)$ is combined with each share of stock for some number A. The new price of a share is $P_{jt}\,A$. Since a bet of Q_{jt} shares transfers the same economic risk, the number of shares in a bet Q_{jt} does not change, and trading volume V_{jt} does not change. Since the economic risk of a bet does not change and trading cash is costless, the dollar cost C of executing the bundle bet does not change either. Each share continues to have the same dollar risk $P_{jt}\,\sigma_{jt}$; therefore, the return standard deviation σ_{jt} changes to σ_{jt}/A, and the return variance σ^2_{jt} changes to σ^2_{jt}/A^2. It is straightforward to verify that

L_{jt} changes to $L_{jt}\,A$ and Z_{jt} remains unchanged. Strategically incorporating the exponent 1/3 into the definition of L_{jt} in equation (4) has the effect of making L_{jt} scale proportionally with A, just like price P_{jt}; if we did not incorporate this exponent of 1/3 into our definition of liquidity, it would show up in subsequent formulas derived from assuming leverage neutrality. Thus, L_{jt} is dimensionless but not leverage neutral, and Z_{jt} is both dimensionless and leverage neutral.

The percentage cost G_{jt} of executing a bet of Q_{jt} shares changes by a factor $1/A$ because the dollar cost of executing this bundled bet remains unchanged while the dollar value of the bundled bet scales proportionally with price, from $P_{jt}\,|Q_{jt}|$ to $P_{jt}\,|Q_{jt}|\,A$. These transformations can be summarized as

$$\begin{aligned} Q_{jt} &\to Q_{jt}, & L_{jt} &\to L_{jt}\,A,\\ V_{jt} &\to V_{jt}, & Z_{jt} &\to Z_{jt},\\ P_{jt} &\to P_{jt}\,A, & C &\to C,\\ \sigma^2_{jt} &\to \sigma^2_{jt}\,A^{-2}, & G_{jt} &\to G_{jt}\,A^{-1}. \end{aligned}$$

For example, let each share of a stock be combined with an equal amount of cash, implying $A = 2$. Then bet size in shares, volume in shares, dollar volatility per share, and the dollar costs of executing a bet stay the same. Both share price and dollar bet size double, percentage returns volatility decreases by a factor of 2, liquidity increases by a factor of 2, and percentage market impact decreases by a factor of 2.

Alternatively, leverage neutrality can be understood as applying Modigliani–Miller equivalence to market microstructure.

ASSUMPTION 3 (Leverage Neutrality II): *If a firm's debt is riskless, then making a change in leverage—the ratio of a firm's debt to its equity—does not change the economic costs associated with trading the firm's securities.*

Suppose the stock is levered up by a factor $1/(1-A)$ as a result of paying a cash dividend of $(1-A)\,P_{jt}$ financed with cash or riskless debt. Since a bet of Q_{jt} shares transfers the same economic risk, the number of shares in a bet Q_{jt} does not change, and trading volume V_{jt} does not change. The dollar cost of the bet C does not change either. The ex-dividend price of a share is $A\,P_{jt}$ because the value of the share-plus-dividend is conserved. Each share continues to have the same dollar risk $P_{jt}\,\sigma_{jt}$; therefore, the return standard deviation σ_{jt} increases to $A^{-1}\,\sigma_{jt}$, and the return variance σ^2_{jt} increases to $A^{-2}\,\sigma^2_{jt}$. It is straightforward to verify that L_{jt} changes to $A\,L_{jt}$ and Z_{jt} remains unchanged. The percentage cost G_{jt} of executing a bet of Q_{jt} shares changes by a factor A^{-1} because the dollar cost of executing this bet remains unchanged while the dollar value of the bet scales inversely proportionally with P_{jt}, from $P_{jt}\,|Q_{jt}|$ to $A\,P_{jt}\,|Q_{jt}|$. These transformations are equivalent to the transformations described above.

For example, if a company levers its stock up by a factor of $A^{-1} = 2$ by paying a cash dividend equal to a half of its size, then the share price drops by a factor of

2, but the economics behind trading the new security does not change. Bet size in shares, volume in shares, dollar volatility per share, and dollar costs of executing bets remain the same. This implies that dollar bet size drops by a factor of 2, percentage returns volatility increases by a factor of 2, liquidity drops by a factor of 2, and percentage market impact increases by a factor of 2.

Finally, leverage neutrality can be interpreted in economic terms as irrelevance of margin requirements and repo haircuts for the economics of trading. Exchanges often require market participants to post cash-equivalent assets into margin accounts; for example, if margin requirements are 20 percent, then traders have to set aside cash-equivalent assets in amounts equal to 20 percent of a transaction while borrowing the remaining 80 percent. Also, risky securities can be used as a collateral to borrow cash; for example, if a repo haircut is 20 percent, then traders can borrow cash in amounts equal to 80 percent of the value of their position. Leverage neutrality implies that changes in margin requirements or repo haircuts do not affect the economic outcomes of trading securities. To illustrate, regardless of whether margin requirements are 20 percent ($A = 0.20$) or 50 percent ($A = 0.50$) and regardless of whether repo haircuts are 10 percent ($A = 0.10$) or 20 percent ($A = 0.20$), the dollar costs of trading securities do not change. Of course, in a more general model, restrictions on margin lending and regulation of repo haircuts may have an equilibrium effect on trading volume and volatility.

ASSUMPTION 4 (Leverage Neutrality III): *Changes in repo haircuts and margin requirements do not change the economic costs of trading risky securities.*

Leverage neutrality essentially imposes one more restriction on the transaction cost formula. Leverage neutrality implies that for any A, the function g satisfies the homogeneity condition

$$g(A\,L_{jt}, Z_{jt}) = A^{-1}\,g(L_{jt}, Z_{jt}). \tag{6}$$

Letting $A = L_{jt}^{-1}$, the function g can be written $g(L_{jt}, Z_{jt}) = L_{jt}^{-1}\,g(1, Z_{jt})$. Define the univariate function f by $f(Z_{jt}) := g(1, Z_{jt})$. Now G_{jt} can be written in the simpler form $G_{jt} = L_{jt}^{-1}\,f(Z_{jt})$: the percentage cost of executing a bet scales inversely with liquidity L_{jt}.

PROPOSITION 2: *If the market impact function $G_{jt} := g()$ is correctly specified as a function the number of shares Q_{jt}, the stock price P_{jt}, share volume V_{jt}, return variance σ_{jt}^2, and dollar bet cost C, then dimensional analysis and leverage neutrality together imply that the function of five parameters simplifies to*

$$G_{jt} := g(Q_{jt}, P_{jt}, V_{jt}, \sigma_{jt}^2, C) = g(L_{jt}, Z_{jt}) = \frac{1}{L_{jt}}\,f(Z_{jt}), \tag{7}$$

where the dimensionless scalar argument L_{jt} and the dimensionless, leverage-neutral scalar argument Z_{jt} are defined in equation (4).

In terms of the original five parameters, the restrictions imposed by dimensional analysis and leverage neutrality in equation (7) can equivalently be spelled out as (8)

$$G_{jt} = g(Q_{jt}, P_{jt}, V_{jt}, \sigma^2_{jt}, C) = \left(\frac{\sigma^2_{jt}\, C}{m^2\, P_{jt}\, V_{jt}}\right)^{1/3} f\left(\left(\frac{\sigma^2_{jt}\, C}{m^2\, P_{jt}\, V_{jt}}\right)^{1/3} \frac{P_{jt}\, Q_{jt}}{C}\right).$$

This general specification is dimensionally homogeneous because, like G, both the argument of f and the constant multiplying it are dimensionless. While equation (8) is consistent with different assumptions about the shape of the function f, neither dimensional analysis nor leverage neutrality says anything about the functional form of f. This must be discovered by solving an economic problem theoretically or by extensive empirical analysis.

Our procedure can be summarized as follows. Conjecture that G_{jt} depends only on the five parameters Q_{jt}, P_{jt}, V_{jt}, σ^2_{jt}, and C. Scale G_{jt} to convert it into dimensionless and leverage-neutral quantity $G_{jt}L_{jt}$. Drop three dimensionally independent and complete arguments (P_{jt}, Q_{jt}, and σ^2_{jt}) that span the three dimensions, and drop one more argument due to the constraint generated by leverage neutrality. Choose the remaining argument Z_{jt} so that it is dimensionless and leverage neutral. It follows that the dimensionless and leverage-neutral product $G_{jt}\, L_{jt}$ must be a function $f()$ of the single dimensionless and leverage-neutral argument Z_{jt}. Thus, the percentage transaction cost (7) can be presented as the product of a security-specific measure of illiquidity $1/L_{jt}$ and a function $f(Z_{jt})$ of scaled bet size Z_{jt}. This formula is consistent with both dimensional analysis and leverage neutrality.

This section shows that adding a cash-equivalent asset significantly simplifies the analysis. This is similar in spirit to obtaining a two-fund separation theorem by adding a riskless asset to the standard mean-variance portfolio optimization problem.

III. Economic Interpretation

The dimensionless variables L_{jt} and Z_{jt} can be given intuitive interpretations. Suppose that bet size $\tilde{Q}_{jt}$ is a random variable with $\mathrm{E}\{\tilde{Q}_{jt}\} = 0$. Without loss of generality, choose the scaling constant m^2 in equation (4) such that

$$\mathrm{E}\{|\tilde{Z}_{jt}|\} = 1. \tag{9}$$

This choice gives a convenient interpretation to the dimensional variables L_{jt} and Z_{jt}. The variable Z_{jt} can be interpreted as *scaled bet size* because it expresses the size of a bet Q_{jt} as a multiple of mean unsigned bet size $\mathrm{E}\{|\tilde{Q}_{jt}|\}$,

$$Z_{jt} = \frac{Z_{jt}}{\mathrm{E}\{|\tilde{Z}_{jt}|\}} = \frac{Q_{jt}}{\mathrm{E}\{|\tilde{Q}_{jt}|\}}. \tag{10}$$

Given the choice of a scaling constant m, the definition of Z_{jt} also implies

$$\frac{1}{L_{jt}} = \frac{C}{\mathrm{E}\{P_{jt}\,|\tilde{Q}_{jt}|\}}. \tag{11}$$

Since the numerator C is the expected dollar cost of a bet (equation (2)) and the denominator $\mathrm{E}\{P_{jt}\,|\tilde{Q}_{jt}|\}$ is the expected dollar value of the bet, the variable $1/L_{jt}$ thus measures the value-weighted expected market impact cost of a bet, expressed as a fraction of the dollar value traded. It is reasonable to interpret $1/L_{jt}$ as an *illiquidity index* and L_{jt} as a *liquidity index*.

PROPOSITION 3: *More liquid markets are associated with more bets of larger sizes. The number of bets γ_{jt} increases with liquidity L_{jt} twice as fast as their sizes $P_{jt}\,|\tilde{Q}_{jt}|$:*

$$\mathrm{E}\{P_{jt}\,|\tilde{Q}_{jt}|\} = C\,L_{jt}. \tag{12}$$

$$\gamma_{jt} = \frac{\sigma_{jt}^2\,L_{jt}^2}{m^2}. \tag{13}$$

PROOF: The choice of a constant m^2 such as $\mathrm{E}\{|\tilde{Z}_{jt}|\} = 1$ implies equation (12). It can be further shown using the definitions of L_{jt} and Z_{jt} in equations (4) that the number of bets, denoted γ_{jt} and equal to $\gamma_{jt} = V_{jt}/\mathrm{E}\{|\tilde{Q}_{jt}|\}$, is given by equation (13). □

Holding volatility σ_{jt}^2 constant, bet size $\mathrm{E}\{P_{jt}\,|\tilde{Q}_{jt}|\}$ is proportional to the liquidity index L_{jt}, and the number of bets γ_{jt} is proportional to the squared liquidity index L_{jt}^2. Equations (12) and (13) impose particular restrictions on the number of bets and their sizes in markets with different levels of liquidity. These important restrictions determine the composition of order imbalances, their standard deviations, and therefore market impact.

Equation (13) also implies another interpretation of the illiquidity measure $1/L_{jt}$. Consider financial markets as operating not in calendar time but rather in *business time*, with a clock linked to the arrival rates of bets γ_{jt}. Then equation (13) implies that the illiquidity measure is proportional to return volatility in business time:

$$\frac{1}{L_{jt}} = \frac{1}{m}\,\frac{\sigma_{jt}}{\gamma_{jt}^{1/2}}. \tag{14}$$

Bets of different sizes arrive to the market at rate γ_{jt} bets per day and move prices by about $\sigma_{jt}/\gamma_{jt}^{1/2}$ per bet, ultimately generating return variance of σ_{jt}^2 per day. The quantity $1/L_{jt}$ scales price impact.

The variable m is implicitly determined by equations (4) and (9). Indeed, suppose we add a volatility condition, which says that all returns variance results from the martingale price impact of bets, $\mathrm{E}\{G_{jt}^2\} = \sigma_{jt}^2/\gamma_{jt}$; this equation further

helps to establish connection between price impact and returns volatility. Plugging equation (7), we obtain $\mathrm{E}\{f^2(\tilde{Z}_{jt})\}/L_{jt}^2 = \sigma_{jt}^2/\gamma_{jt}$. Using equation (13), we obtain $m^2 = \mathrm{E}\{f^2(\tilde{Z}_{jt})\}$. If the function f is a power function $f(Z) = \bar{\lambda}\, Z^{\omega}$, then $m^2 = \bar{\lambda}^2\, \mathrm{E}\{\tilde{Z}_{jt}^{2\omega}\}$. Then we obtain the moment ratio equation

$$m^2 = \mathrm{E}\left\{f^2(\tilde{Z}_{jt})\right\} = \frac{\mathrm{E}\{\tilde{Z}_{jt}^{2\omega}\}}{(\mathrm{E}\{\tilde{Z}_{jt}^{1+\omega}\})^2} \tag{15}$$

since plugging equations (4) and (7) into equation (2) imply $\bar{\lambda}\,\mathrm{E}\{\tilde{Z}_{jt}^{1+\omega}\} = 1$. If $\omega = 1$, then

$$m = \frac{\mathrm{E}\{(|\tilde{Z}_{jt}|\})}{(\mathrm{E}\{\tilde{Z}_{jt}^2\})^{1/2}} = \frac{\mathrm{E}\{(|\tilde{Q}_{jt}|\})}{(\mathrm{E}\{\tilde{Q}_{jt}^2\})^{1/2}}. \tag{16}$$

We refer to m as a *moment ratio* for the distribution of bet sizes Q_{jt} or scaled bet sizes Z_{jt}.

IV. Market Microstructure Invariance.

Dimensional analysis does not generate operational market microstructure predictions per se. To obtain useful empirical predictions based on transaction cost model (8), it is necessary to think about how to measure relevant quantities. The derivation above refers to at least five quantities: asset price P_{jt}, trading volume V_{jt}, return volatility σ_{jt}, bet size Q_{jt}, bet cost C, and possibly other measures of transaction costs such as bid-ask spreads. Three of the quantities—asset price P_{jt}, trading volume V_{jt}, and return volatility σ_{jt}—can be observed directly or readily estimated from public data on securities transactions; these are observable characteristics of an asset. The size Q_{jt} is a characteristic of a bet privately known to a trader. While bid-ask spreads can be observed from public data, other estimates of transaction costs generally require having confidential data on transactions which allow transactions of one trader to be distinguished from transactions of another. More ambiguous is the issue of how the cost of a bet C and the scaling parameter m^2 might or might not vary across assets.

In the context of this paper, *market microstructure invariance* is defined as the following two empirical hypotheses:

ASSUMPTION 5 (Market Microstructure Invariance): *The dollar value of C and the dimensionless scaling parameter m^2 are the same for all time periods and for all assets such as stocks, bonds, commodities, foreign exchange, and derivatives.*

These invariance hypotheses are neither implications of dimensional analysis nor implications of leverage neutrality. The a priori justification for the invariance hypotheses is Ockham's razor: these are the simplest possible empirical hypotheses. The invariance hypothesis about C is motivated by the intuition that asset managers allocate scarce intellectual resources across assets and across time in such a

manner that the dollar cost of bets C is equated. Kyle and Obizhaeva (2016*b*), who derive equivalent invariance principles as endogenous properties of a dynamic equilibrium model of informed trading, show that this assumption is also related to the economic intuition that traders are indifferent between spending resources on generating trading signals in different markets. The scalar m^2 is expected to be constant across assets and across time if distributions of informational signals and therefore bet sizes are constants across assets and across time.

To apply invariance across markets with different currencies and real exchange rates, it is necessary to scale the expected dollar cost C by the productivity-adjusted wages of finance professionals in the local currency. This additional scaling makes the parameter C dimensionless by adjusting for inflation and productivity. For example, it is possible that the bet cost C is not the same in Russia and the United States if the wages of finance professionals are different in the two countries due to differences in productivity, inflation, or real and nominal exchange rates. One can then assume that C is proportional to productivity-adjusted nominal wages. If a finance professional's productivity is measured as number of bets processed per day, denoted b, and the nominal wage for finance professionals per day is w, then $C = c\,w/b$, where C, w, and b may vary across countries, but c can be conjectured to be an invariant dimensionless constant. By examining the Russian and the U.S. markets separately, we implicitly assume that C may vary across countries but is the same within a country.

Under the invariance assumptions, instead of having different models for different securities and different time periods, it is necessary to calibrate only two parameters C and m^2 for all assets, not different values for each asset. Together with the shape of the invariant cost function $f(Z_{jt})$ in equation (7), the knowledge of the parameters C and m^2 makes it possible to write a universal operational transaction cost model for all markets. The constants C and m^2 essentially help to relate the microscopic details of trading in a security to its macroscopic properties. Specifically, these two invariant constants relate the microscopic size distribution and transaction cost of a bet to observable macroscopic dollar volume $P_{jt}\,V_{jt}$ and volatility σ_{jt}.

Preliminary calibration in Kyle and Obizhaeva (2016*c*) is consistent with the approximate values $C \approx \$2000$ and $m^2 \approx 0.25$. These estimates are based on a large sample of portfolio transitions orders. A portfolio transition occurs when a large investor, such as a pension plan sponsor, hires a professional third party to make the trades necessary to move assets from one asset manager to another. In what follows, we drop C and m^2 from our formulas for simplicity of exposition.

If invariance holds, the dimensionless liquidity index $L_{jt} \sim \left(P_{jt}\,V_{jt}\,\sigma_{jt}^{-2}\right)^{1/3}$ is a natural, simple measure of liquidity which is easy to calculate using data on volume and volatility. These security-specific metrics do not change when a stock splits or the frequency with which data is sampled changes. Plugging calibrated numerical

values for C and m^2 into equation (4) yields a specific formula for $1/L_{jt}$:

$$1/L_{jt} = 20 \left(\frac{\sigma_{jt}^2 \, \$1}{P_{jt} \, V_{jt}} \right)^{1/3}. \tag{17}$$

This is our main illiquidity index.

Equation (8) is a general structural transaction cost model. The general specification f for a transaction cost function (8) is consistent with different functional forms. Next, we present several of them.

PROPOSITION 4: *Suppose the market microstructure invariance assumptions hold (Assumption 5) and function f in equation (7) is a power function of the form $f(Z_{jt}) = \bar{\lambda} \, |Z_{jt}|^{\omega}$. Then, a proportional bid-ask spread cost ($\omega = 0$) implies*

$$G_{jt} = \text{const} \cdot \frac{1}{L_{jt}}. \tag{18}$$

A linear market impact cost ($\omega = 1$) implies

$$G_{jt} = \text{const} \cdot \frac{P_{jt} \, |Q_{jt}|}{C \, L_{jt}^2}. \tag{19}$$

A square-root market impact cost ($\omega = 1/2$) implies

$$G_{jt} = \text{const} \cdot \sigma_{jt} \left(\frac{|Q_{jt}|}{V_{jt}} \right)^{1/2}. \tag{20}$$

PROOF: We obtain these formulas by plugging Z_{jt} from equation (4) and $f(Z_{jt}) = \bar{\lambda} \, |Z_{jt}|^{\omega}$ with different exponents ω into equation (7) and then collecting all constant terms. □

Equations (18), (19), and (20) are special cases consistent with invariance. In the three transaction cost models (18), (19), and (20), the constants on the right side are dimensionless.

The proportional market impact model (18) suggests a formula for a bid-ask spread costs, because the exponent $\omega = 0$ implies that the transaction cost G_{jt} is a constant percentage of the value of the asset, which does not depend on the size of the bet. Bid-ask spreads are predicted to be inversely proportional to the liquidity measure with a proportionality constant which is the same for all assets. The linear price impact model (19) implements the price impact parameter $\lambda = \sigma_V / \sigma_U$ from the model of Kyle (1985) as $\lambda_{jt} = \text{const} \cdot (P_{jt}^2 / C) \cdot (1/L_{jt}^2)$. Linear impact is consistent with many theoretical models of speculative trading with adverse selection. Empirical estimates often support the square root specification (20), as noted in Grinold and Kahn (1995).

The power law is a convenient functional form for market impact because it nests many important cases including bid-ask spreads, linear impact, and the square root

model. The empirical literature sometimes combines them together by considering, for example, the sum of a bid-ask spread cost and a linear impact cost. A generalization of the Stone-Weierstrass theorem implies that any continuous function can be approximated as a linear combination of power functions (uniformly on compact sets).

V. Empirical Evidence

This section uses data from Russian and U.S. equity markets to examine the predictions of market microstructure invariance for the size of quoted bid-ask spreads and the numbers of trades. While quoted spreads are not exactly the same as realized spread costs and trades are not the exactly the same as bets, dimensional analysis, leverage neutrality, and microstructure invariance have the same implications for quoted spreads and trades as for spread costs and bets.

Dimensional analysis and leverage neutrality imply a scaling law for the quoted bid-ask spread. Let S_{jt} denote the bid-ask spread, measured in the same units as price P_{jt}. It measures a constant proportional component of transaction costs obtained by setting $\omega = 0$ (equation (18)). Thus, market microstructure invariance implies

$$\ln\left(\frac{S_{jt}}{P_{jt}}\right) = \text{const} + 1 \cdot \ln\left(\frac{1}{L_{jt}}\right). \tag{21}$$

For empirical estimation, the unknown invariant constants C and m^2 can be factored out of the definition of L_{jt} and incorporated into the constant term in equation (21). From the definition of $1/L_{jt}$, the coefficient of one on $\ln(1/L_{jt})$ implies a scaling exponent of $-1/3$ on $P_{jt}\, V_{jt}\, \sigma_{jt}^{-2}$.

We also present results of testing a scaling relationship for the number of bets. Bets are difficult to observe, since they are typically executed in the market as many trades and shared by several traders. Let N_{jt} denote the number of trades which occur per calendar day. If institutional microstructure details such as tick size and minimum lot size adjust across stocks to have similar effects on trading, it is reasonable to conjecture that the number of trades N_{jt} is proportional to the number of bets γ_{jt}. Then, from equation (13), market microstructure invariance implies

$$\ln\left(N_{jt}\right) = \text{const} + 2 \cdot \ln(\sigma_{jt} L_{jt}). \tag{22}$$

To test these relationships, we use two datasets. First, we use data from the Moscow Exchange from January to December 2015 provided by Interfax Ltd. The data cover the 50 Russian stocks in the RTS Index ("Russia Trading System") as of June 15, 2015. The five largest companies are Gazprom, Rosneft, Lukoil, Novatek, and Sberbank. The Russian stock market is centralized with all trading implemented in a consolidated limit-order book. The tick size is regularly adjusted by exchange officials. The lot size is usually small. For each of the 50 stocks and

each of the 250 trading days, the average percentage spread is calculated as the mean of the percentage spread at the end of each minute during trading hours from 10:00 to 18:50. The realized volatility is calculated based on summing squared one-minute changes in the mid-point between the best bid and best offer prices at the end of each minute during trading hours. Table 1 presents summary statistics for the Russian sample.

		Units	Avg	p5	p50	p95	p100
Cap_{jt}	Mkt. Cap.	USD Billion	8.76	0.44	3.49	35.02	62.92
Cap_{jt}	Mkt. Cap.	RUB Billion	476	24	190	1904	3420
$V_{jt}\,P_{jt}$	Volume	RUB Million/Day	542	3	73	2607	6440
σ_{jt}	Volatility	$10^{-4}/\text{Day}^{1/2}$	189	130	180	260	290
S_{jt}/P_{jt}	Spread	10^{-4}	20	4	13	66	131
N_{jt}	Trades	Count/Day	7328	65	2792	22 169	71 960

Table 1—The table presents summary statistics (average values and percentiles) for the sample of 50 Russian stocks: dollar and ruble capitalization Cap_{jt} (in billions), average daily volume $V_{jt}\,P_{jt}$ in millions of rubles, daily return volatility σ_{jt}, average percentage spread S_{jt}/P_{jt} in basis points, and average number of trades per day N_{jt} as of June 2015.

Second, we also use daily TAQ (Trade and Quote) data for U.S. stocks from January to December 2015. The data cover 500 stocks in the S&P 500 index as of June 15, 2015. The largest companies are Apple, Microsoft, and Exxon Mobil. The U.S. stock market is highly fragmented, with securities traded simultaneously on dozens of exchanges. For most securities, the minimum tick size is equal to one cent ($0.01), which may be binding for stocks with low price or high dollar volume. The minimum lot size is usually 100 shares. For each of the 500 stocks and each of the 252 trading days, the average percentage spread is calculated as the mean of the percentage spread, based on the best bids and best offers across all exchanges, at the end of each minute during the hours from 9:30 to 16:00. The realized volatility is calculated based on summing squared one-minute changes in the mid-point between the best bid and best offer prices at the end of each minute during trading hours. Table 2 presents summary statistics for the U.S. sample.

Figure 1 plots the log bid-ask spread $\ln(S_{jt}/P_{jt})$ against $\ln(1/L_{jt})$ using the data from the Moscow Exchange. Each of 12 426 points represents the average bid-ask spread for each of 50 stocks in the RTS index for each of 250 days. Different colors represent different stocks. For comparison, we add a solid line $\ln(S_{jt}/P_{jt}) = 2.112 + 1\,\ln(1/L_{jt})$, where the slope of one is fixed at the level predicted by market microstructure invariance and the intercept is estimated. All observations cluster around this benchmark line.

In the aggregate sample, the fitted line is $\ln(S_{jt}/P_{jt}) = 2.093 + 0.998\,\ln(1/L_{jt})$, with standard errors of estimates clustered at daily levels equal to 0.040 and 0.005,

			Avg	p5	p50	p95	p100
Cap_{jt}	Mkt. Cap.	USD Billion	38	6	18	160	716
$V_{jt}\,P_{jt}$	Volume	USD Million	205	38	124	591	4647
σ_{jt}	Volatility	$10^{-4}/\text{Day}^{1/2}$	110	70	90	160	4050
S_{jt}/P_{jt}	Spread	10^{-4}	4	2	3	8	184
N_{jt}	Trades	1/Day	21 048	5308	15 735	58 938	190 270

Table 2—The table presents summary statistics (average values and percentiles) for the sample of 500 U.S. stocks: dollar capitalization Cap_{jt} (in billions), average daily volume $V_{jt}\,P_{jt}$ in millions of dollars, daily return volatility σ_{jt}, average percentage spread S_{jt}/P_{jt} in basis points, and average number of trades per day N_{jt} as of June 2015.

respectively; the R-square is 0.876. The invariance prediction that the slope coefficient is one is not statistically rejected. The fitted line for a similar regression over monthly averages instead of daily averages is $\ln(S_{jt}/P_{jt}) = 2.817 + 1.078\ \ln(1/L_{jt})$ with standard errors of estimates 0.164 and 0.019, respectively; its R-square is 0.923. The invariance prediction that the slope coefficient is one is statistically rejected in this case, but remains economically close to the predicted value.

The 50 dashed lines in figure 1 are fitted based on data for the 50 Russian stocks. The slopes for individual stocks, which vary from 0.249 to 1.011, are substantially lower than the invariance-implied slope of one, which is indistinguishable from the fitted line for the aggregate data. Most slope estimates are close to 0.70; ten slope estimates are less than 0.50, and six slope estimates are larger than 0.90.

Figure 2 plots the log bid-ask spread $\ln(S_{jt}/P_{jt})$ against $\ln(1/L_{jt})$ for the U.S. stocks. Each of 124 170 points depicts the average bid-ask spread for each of the 500 stocks in the S&P 500 index and for each of 252 days. As before, different colors represent different stocks. We also add a solid line $\ln(S_{jt}/P_{jt}) = 1.371{+}1\ \ln(1/L_{jt})$, where the slope of one is fixed at the level predicted by market microstructure invariance and only the intercept is estimated. The intercept of 1.371 for this sample is smaller than the intercept of 2.112 for the sample of Russian stocks. All observations again cluster along the benchmark line, even though there are some visible outliers. The points appear to be more dispersed than in the previous figure.

The fitted line is $\ln(S_{jt}/P_{jt}) = 1.011 + 0.961\ \ln(1/L_{jt})$ with clustered standard errors of estimates being equal to 0.225 and 0.024, respectively, in the aggregate sample; the R-square is 0.450. The invariance prediction that the slope coefficient is one is not statistically rejected. Depicted with dashed lines, the slopes of 500 fitted lines for 500 individual stocks range from -0.052 to 1.667. Most of 500 slope estimates lie between 0.90 and 1.30; about 50 stocks have the slope estimates below 0.50 with the three of them being very close to zero, and about 20 of stocks have slope estimates higher then 1.50.

The fact that both regressions have a slope close to the predicted value of one in-

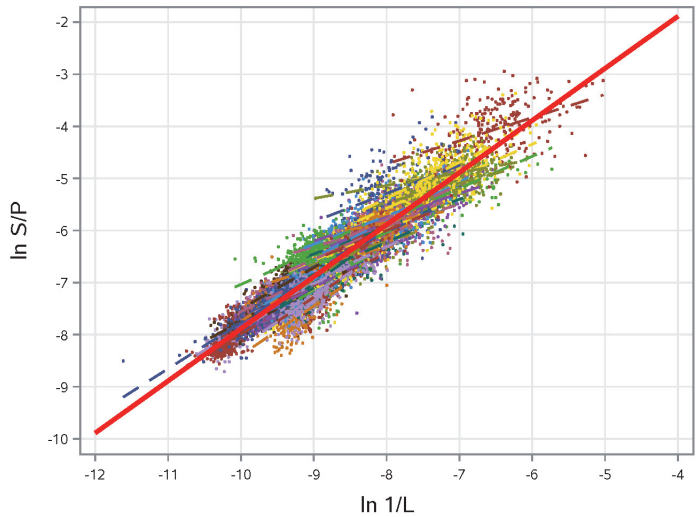

FIGURE 1. THIS TABLE PLOTS THE BID-ASK SPREAD $\ln S/P$ AGAINST ILLIQUIDITY $\ln 1/L$, WITH $1/L = (PV/\sigma^2)^{-1/3}$ FOR EACH OF THE 50 RUSSIAN STOCKS IN THE RTS INDEX FOR EACH OF 250 DAYS FROM JANUARY TO DECEMBER 2015.

dicates that stock markets in both countries adjust over time so the the invariance relationships hold as approximations. In both countries, there are economically significant deviations from invariance in the sense that the R^2 of the regressions is less than one. Deviations from invariance may have different institutional explanations in each country related to minimum tick size and minimum lot size. In the Russian market, frequent non-uniform adjustments of tick sizes may reduce distortions associated with tick size restrictions. In the U.S. market, tick size is fixed at the same level of one cent for most securities, but the tick size as a fraction of the stock price varies when the stock price changes as a result of market movement or stock splits. The typical large company has a higher stock price than the typical small company. Stock splits in response to market movements imply a very slow process for tick size adjustment, and this may lead to more noise in the invariance relationship estimated for U.S. stocks.

Figure 3 presents results of testing the invariance prediction for the number of trades from equation (22) using data from the Moscow Exchange. The figure has 12 426 points plotting the log number of transactions $\ln(N_{jt})$ against $\ln(\sigma_{jt}L_{jt})$ for each of 50 stocks and each of 250 days. For comparison with the prediction of invariance, a benchmark line $\ln(N_{jt}) = -1.937 + 2\ln(\sigma_{jt}L_{jt})$ is added; this line has a slope that is fixed at the predicted level of two and an intercept that is estimated. The results for the aggregate sample are broadly consistent with the predictions. The fitted line is $\ln(N_{jt}) = -3.085 + 2.239\ln(\sigma_{jt}L_{jt})$ with standard errors of estimates equal to 0.038 and 0.008, respectively; its R-square is 0.882.

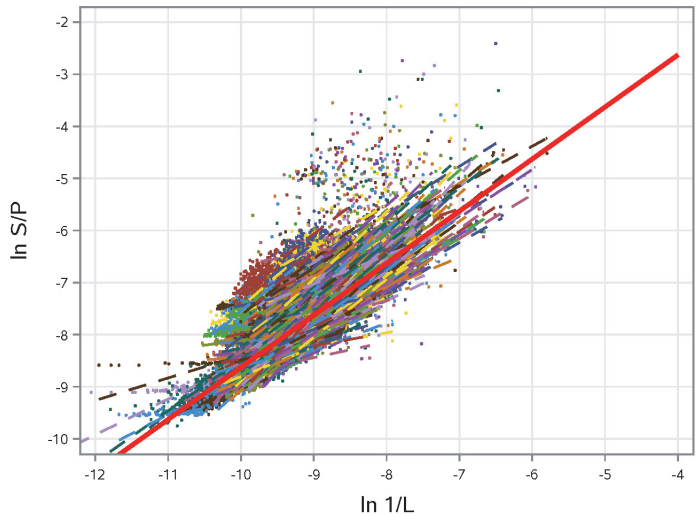

FIGURE 2. THIS TABLE PLOTS THE BID-ASK SPREAD $\ln S/P$ AGAINST ILLIQUIDITY $\ln 1/L$, WITH $1/L = (PV/\sigma^2)^{-1/3}$ FOR EACH OF THE 500 U.S. STOCKS IN THE S&P 500 INDEX FOR EACH OF 252 DAYS FROM JANUARY TO DECEMBER 2015.

As before, the slopes of fitted lines for individual stocks are systematically lower, ranging from 1.156 to 1.795 and depicted with dashed lines.

Figure 4 presents results of testing similar prediction (13) for the number of trades N_{jt} using the data for the U.S. stocks in the S&P 500 index. The figure has 121 760 points plotting the log number of transactions $\ln(N_{jt})$ against $\ln(\sigma_{jt}L_{jt})$ for each of 500 U.S. stocks and each of 252 days. For comparison with the prediction of invariance, a benchmark line $\ln(N_{jt}) = 0.251 + 2\ \ln(\sigma_{jt}L_{jt})$ is added; this line has a slope that is fixed at the predicted level of two and an intercept that is estimated. The intercept of 0.251 for the U.S. benchmark line is higher than the intercept of -1.937 for the Russian benchmark line. Thus, there are approximately $e^{0.251+1.937}$ or nine times more transactions in the highly fragmented U.S. equity market, possibly reflecting numerous cross-market arbitrage trades between different trading platforms. The results for the aggregate sample are broadly consistent with the predicted slope of 2. The fitted line is $\ln(N_{jt}) = 1.005 + 1.842\ \ln(\sigma_{jt}L_{jt})$ with standard errors of estimates equal to 0.054 and 0.011, respectively; its R-square is 0.702.

The slopes of fitted lines for individual stocks, depicted as before with dashed lines, are systematically lower than predicted, ranging from 0.416 to 2.646 and clustering mostly between levels of 1.50 and 1.70. The intercepts of the fitted lines for individual stocks also vary substantially, even though basic invariance hypotheses predict that all intercepts must be the same.

We offer several explanations for why the slopes for individual securities are different from the slopes for the aggregate sample and the slopes predicted by in-

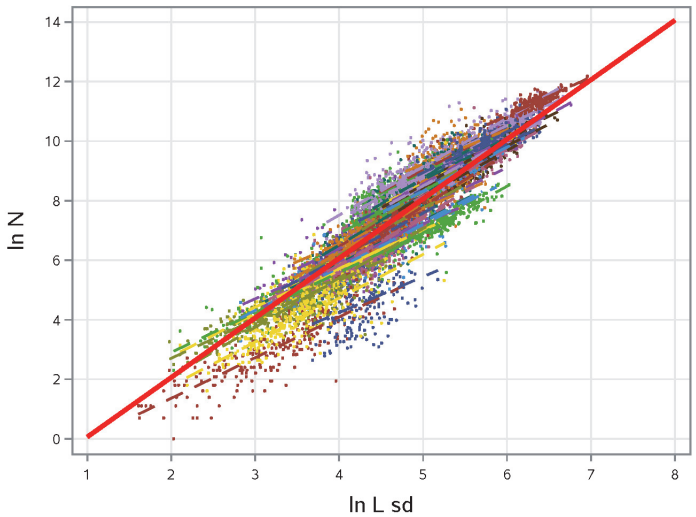

FIGURE 3. THIS TABLE PLOTS THE NUMBER OF TRADES $\ln N$ AGAINST LIQUIDITY $\ln \sigma L$ SCALED BY VOLATILITY σ, WITH $L = (PV/\sigma^2)^{1/3}$ FOR EACH OF THE 50 RUSSIAN STOCKS IN THE RTS INDEX FOR EACH OF 250 DAYS FROM JANUARY TO DECEMBER 2015.

variance. They may be related to a combination of econometric issues, economic issues, or conceptual issues. Testing different hypotheses and assessing their relative importance is an interesting topic which takes us beyond the scope of this paper.

First, it is possible that a substantial part of the variation in stock-specific measures of liquidity on the right-hand side of the regression equations is due to variations in liquidity of the overall market and therefore may not reflect variations in bid-ask spreads or number of transactions of individual securities. The existence of noise in regressors may bias slope estimates downwards.

Second, estimates may be biased due to likely correlation between explanatory variables and error terms. For example, execution of large transactions will mechanically be reflected in larger volume—and thus a higher liquidity measure—as well as a larger bid-ask spread, since they are often executed against existing limit orders and liquidity is not replenished instantaneously.

Third—and perhaps most importantly—some discrepancies may be explained by differences in how market frictions such as minimum lot size and minimum tick size affect bid-ask spreads and trade size. We outline in Section VII a conceptual approach for adjusting predictions for these frictions.

VI. Methodological Issues

This section re-examines Assumption 1, which states that $g(\ldots)$ is correctly specified as a function of five specific parameters. We first examine whether unneces-

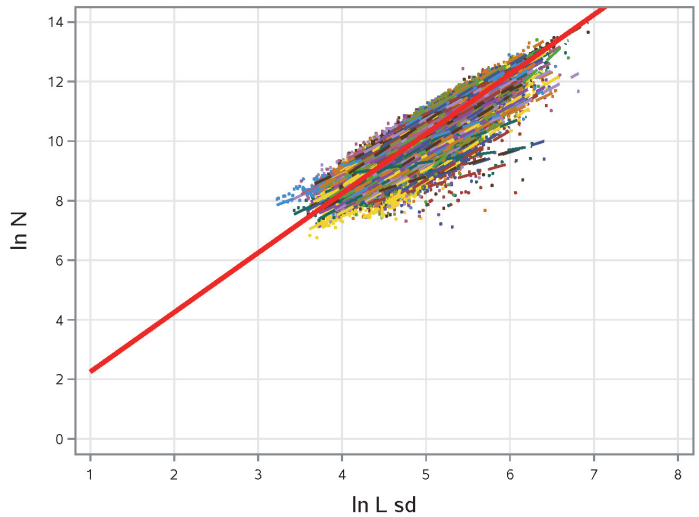

FIGURE 4. THIS TABLE PLOTS THE NUMBER OF TRADES $\ln N$ AGAINST LIQUIDITY $\ln \sigma L$ SCALED BY VOLATILITY σ, WITH $L = (PV/\sigma^2)^{1/3}$ FOR EACH OF THE 500 U.S. STOCKS IN THE S&P 500 INDEX FOR EACH OF 250 DAYS FROM JANUARY TO DECEMBER 2015.

sary parameters are included in the model. We then examine whether necessary parameters have been omitted.

Can Our Model be Simplified?

Dimensional analysis depends crucially on the set of parameters included. It is possible that we may have initially included some unnecessary parameters in the model. Suppose that the transaction cost function depends only on four of the hypothesized five parameters, $G = g(Q_{jt}, P_{jt}, V_{jt}, \sigma_{jt}^2)$, and not on the non-intuitive bet cost C.

If so, then dimensional analysis implies that G_{jt} is a function of only one argument, $Q_{jt}\, V_{jt}/\sigma_{jt}^2$, which is dimensionless but not leverage neutral. Leverage neutrality further implies that this parameter must be scaled proportionally with leverage, yielding the square root specification equivalent to equation (20):

$$G_{jt} = \text{const} \cdot \sigma_{jt} \left(\frac{|Q_{jt}|}{V_{jt}} \right)^{1/2}. \tag{23}$$

Without C, the simplified specification mandates a square root model. Pohl et al. (2017) formalize this derivation mathematically. Leaving out C thus makes the model very inflexible.

One big advantage of having C in the specification is that it allows nesting linear price impact, bid-ask spreads, and the square root model into one specification governed by the parameter ω. Our original analysis is consistent with equilibrium

models that imply a linear market impact.

Another subtle but powerful argument favors inclusion of C into the list of arguments. Suppose that one would like to derive a model for the distribution of bet sizes Q_{jt} under the assumption that it may also depend only on the three parameters P_{jt}, V_{jt}, and σ^2_{jt}, but not C. Then dimensional analysis implies that the distributions of scaled bet sizes $Q_{jt}\,\sigma^2_{jt}/V_{jt}$ are invariant. Since the scaled variable is not leverage neutral, it makes this specification inconsistent with the leverage neutrality assumption. In the last section, we show that inclusion of C into the list of arguments for bet sizes makes our model more flexible and allows us to circumvent this problem.

One might think that if the value of C does not vary in all applications of interest, then this parameter should be dropped from the list of arguments. On the contrary, dimensional analysis must be based on a complete set of arguments, even though values of some of them are fixed. Simply omitting these variables, even constant ones, leads to erroneous results. The correct simplification algorithm is to replace the set of all fixed parameters with a dimensionally independent subset of themselves and then redo the dimensional analysis, as described in Sonin (2001). Thus, although the value of C may be fixed in all applications of interest, it should not be excluded from the original list of parameters since it represents a dimensionally independent subset of itself. In practice, whether values of bet costs C are indeed fixed remains an empirical question. If C varies across countries or time periods, then this variable may possibly determine similarity groups across markets, similar to Reynolds numbers in the turbulence theory.

Sometimes parameters can be eliminated by defining new units. For example, Newton's original law of motion says that force is proportional to the product of mass and acceleration, $F \sim M\,a$. If one chooses a force unit such that one unit of force will give unit mass unit acceleration, then the proportionality constant drops out and the equation becomes $F = M\,a$. Similarly, if we introduce a new unit of currency equal to the expected cost of executing a bet (a fundamental unit of money), then C will drop out of all equations. For scientific studies in market microstructure, this would be a natural unit of currency. Note that since the moment ratio m is already dimensionless, it is impossible to eliminate it by redefining units.

Could Some Variables Have Been Omitted?

It is possible that we may have initially excluded some necessary parameters. For example, the predictions may hold most closely when minimum tick size is small, minimum lot size is not restrictive, market makers are competitive, and transaction fees and taxes are minimal. When these assumptions are not met, invariance principles provide a benchmark from which the importance of frictions can be measured.

The empirical implications of dimensional analysis, leverage neutrality, and market microstructure invariance can be generalized to incorporate other variables.

Here is a general methodology for deriving relationships among financial variables, following the Buckingham π-theorem. Suppose we would like to study a variable Y.

- Write down all variables X_1, X_2, X_3,...,X_m that may affect Y.
- Construct a dimensionless and leverage-neutral variable $\alpha_y Y$ from Y by scaling it by a product of powers of X_1,...,X_m with different exponents, $\alpha_y = X_1^{y_1} \cdots X_m^{y_m}$.
- Drop three dimensionally independent arguments used up to make the equation dimensionally consistent and match the dimension of Y, which is made up of the three finance units. Drop one more argument used up to satisfy a leverage neutrality constraint.
- Scale the remaining arguments X_5,...,X_m by a product of powers of X_1,..., X_m with different exponents, $\alpha_i = X_1^{i_1} \cdots X_m^{i_m}$ for $i = 5, \ldots, m$ to construct dimensionless and leverage-neutral variables $\alpha_5 X_5$,...,$\alpha_m X_m$.
- Then the resulting equation for Y is $\alpha_y Y = f(\alpha_5 X_5, \ldots, \alpha_m X_m)$.

This is a generalized algorithm for dimensional analysis and leverage neutrality. It shows how to include any number of additional explanatory variables into the model.

Including unnecessary parameters does not make a model logically incorrect, but it does reduce its statistical power by making it unnecessarily complicated. Each new parameter adds a new variable into a scaling law. If unnecessary variables are included, then extensive empirical analysis is necessary to show that these parameters are unnecessary.

General Scaling Laws for Market Impact Model.

We next derive a more general version of the market impact model (7) that includes three additional variables. Transaction costs may depend on the execution horizon and market frictions such as minimum tick size and minimum lot size. The tick size for U.S. stocks is generally one cent, and the minimum round lot size is generally 100 shares; for Russian stocks there is more variation in these parameters.

First, add to the original five parameters three additional parameters: the horizon of execution T_{jt} measured in units of time, the minimum tick size $K_{jt}^{\min}$ measured in currency per share, and the minimum round lot size $Q_{jt}^{\min}$ measured in shares. Second, re-scale the new explanatory variables T_{jt}, $K_{jt}^{\min}$, and $Q_{jt}^{\min}$ to make them dimensionless and leverage neutral using the four variables P_{jt}, V_{jt}, σ_{jt}^2, and C (including the liquidity variable L_{jt} and the dimensionless moment parameter m). The re-scaled values are $T_{jt}\,\sigma_{jt}^2\,L_{jt}^2/m^2$, $K_{jt}^{\min}\,L_{jt}/P_{jt}$, and $Q_{jt}^{\min}\,\sigma_{jt}^2\,L_{jt}^2/V_{jt}$, respectively (up to constants of proportionality). It is convenient to let B_{jt} denote the

scaled execution horizon. Equation (13) for γ_{jt} yields

$$B_{jt} := \frac{T_{jt}\,\sigma_{jt}^2\,L_{jt}^2}{m^2} = T_{jt}\,\gamma_{jt}. \tag{24}$$

The variable B_{jt} measures the expected number of bets over which a given bet is executed; it converts clock time to business time.

If minimum tick size and minimum lot size do not affect market impact costs, then the equation (7) becomes

$$G_{jt} = \frac{1}{L_{jt}}\,f\left(Z_{jt}, B_{jt}\right), \tag{25}$$

where Z_{jt} is scaled bet size defined in equation (4) and B_{jt} is scaled execution time defined in equation (24).

For example, if price impact is linear in both the size of bets and their rate ofbi execution, then the market impact model becomes

$$G_{jt} = \frac{1}{L_{jt}}\,\left(\lambda\,Z_{jt} + \kappa\,Z_{jt}/B_{jt}\right). \tag{26}$$

Larger bets executed at a faster rate tend to incur higher transaction costs. This specification of price impact is derived endogenously in the dynamic model of speculative trading of Kyle, Obizhaeva and Wang (2017).

More generally, the market impact model (7) generalizes to

$$G_{jt} = \frac{1}{L_{jt}}\,f\left(\frac{P_{jt}\,Q_{jt}}{C\,L_{jt}}, \frac{T_{jt}\,\sigma_{jt}^2\,L_{jt}^2}{m^2}; \frac{K_{jt}^{\min}\,L_{jt}}{P_{jt}}, \frac{Q_{jt}^{\min}\,\sigma_{jt}^2\,L_{jt}^2}{V_{jt}}\right). \tag{27}$$

This specification remains consistent with our scaling laws but allows for non-linear relationships among the different arguments of f. Here, the first two arguments are characteristics of a bet and its execution, and the last two arguments are characteristics of the marketplace. Other variables can be easily added to the transaction cost model following the same algorithm.

VII. Extensions and Other Applications.

Our approach allows us to derive other scaling laws. This flexibility makes it more general than the approach discussed in Kyle and Obizhaeva (2016*c*). Next, we present several extensions. Testing these additional predictions empirically takes us beyond the scope of this paper. We present them as illustrations of promising directions for future research.

Scaling Laws for Optimal Execution Horizon.

Optimal execution horizon is obviously of interest to traders. Suppose that the optimal (cost-minimizing) execution horizon T_{jt}^* or, alternatively, the trading rate

$|Q_{jt}|/T^*_{jt}$ depends on the seven variables Q_{jt}, P_{jt}, V_{jt}, σ^2_{jt}, C, $K^{\min}_{jt}$, and $Q^{\min}_{jt}$. When tick size is large, larger quantities available at the best bid and offer prices may make the execution horizon shorter. Execution horizon may also depend on order size in a non-linear fashion.

Since the ratio $|Q_{jt}|/(V_{jt}\,T^*_{jt})$ is dimensionless and leverage neutral, the same logic as above implies that an optimal execution horizon is consistent with a function t^* of three dimensionless and leverage neutral parameters:

$$\frac{|Q_{jt}|}{V_{jt}\,T^*_{jt}} = t^*\left(\frac{P_{jt}\,Q_{jt}}{C\,L_{jt}}; \frac{K^{\min}_{jt}\,L_{jt}}{P_{jt}}, \frac{Q^{\min}_{jt}\,\sigma^2_{jt}\,L^2_{jt}}{V_{jt}}\right). \tag{28}$$

Our analysis does not allow us to place more restrictions on the function t^*. If tick size and minimum lot size do not affect execution horizon, then the participation rate $|Q_{jt}|/(V_{jt}\,T^*_{jt})$ depends only on the first argument of function t^*, the scaled bet size $Z_{jt} := P_{jt}\,Q_{jt}/(C\,L_{jt})$ from equation (4):

$$\frac{|Q_{jt}|}{V_{jt}\,T^*_{jt}} = t^*\left(\frac{P_{jt}\,Q_{jt}}{C\,L_{jt}}\right). \tag{29}$$

If the function t^* is a constant, then

$$\frac{|Q_{jt}|}{V_{jt}\,T^*_{jt}} = \text{const}. \tag{30}$$

It is optimal to choose the execution horizon so that traders execute all trades targeting the same fraction of volume, say one percent of volume until execution of the bet is completed.

Scaling Laws for Optimal Tick Size and Lot Size.

Setting optimal tick size and minimum lot size is of interest for exchange officials and regulators. Let $K^{\min *}_{jt}$ and $Q^{\min *}_{jt}$ denote optimal tick size and optimal minimum lot size, respectively. Suppose both of them depend on the four variables P_{jt}, V_{jt}, σ^2_{jt}, and C.

Since the scaled optimal quantities $K^{\min *}_{jt}\,L_{jt}/P_{jt}$ and $Q^{\min *}_{jt}\,L^2_{jt}\,\sigma^2_{jt}/V_{jt}$ are dimensionless and leverage neutral, the scaling laws for these market frictions can be written as

$$K^{\min *}_{jt} = \text{const}\cdot\frac{P_{jt}}{L_{jt}}, \qquad Q^{\min *}_{jt} = \text{const}\cdot\frac{V_{jt}}{L^2_{jt}\,\sigma^2_{jt}}. \tag{31}$$

Since the proportionality constants do not vary across securities, these measures provide good benchmarks for comparing how restrictive actual tick size and minimum lot size are for different securities and across markets.

If traders choose execution horizons T^*_{jt} optimally according to equation (28) and exchanges set tick size $K^{\min}_{jt}$ and minimum lot size $Q^{\min}_{jt}$ at their optimal levels (31), then function f in market impact model (27) becomes again a function of only one argument Z_{jt}.

Scaling Laws for Bid-Ask Spread.

Our approach can be also used to derive more general scaling laws for the bid-ask spread. The bid-ask spread is an integer number of ticks which fluctuates as trading occurs. Let S_{jt} denote the average bid-ask spread, measured in dollars per share.

Assume the average bid-ask spread depends on the six variables P_{jt}, V_{jt}, σ_{jt}^2, C, $K_{jt}^{\min}$, and $Q_{jt}^{\min}$. Re-scale the bid-ask spread as $S_{jt}\, L_{jt}/P_{jt}$ to make it dimensionless and leverage neutral. Then dimensional analysis and leverage neutrality imply that it is a function s of the two re-scaled dimensionless and leverage-neutral variables $K_{jt}^{\min}$ and $Q_{jt}^{\min}$:

$$\frac{S_{jt}}{P_{jt}} = \frac{1}{L_{jt}}\, s\left(\frac{K_{jt}^{\min}\, L_{jt}}{P_{jt}}, \frac{Q_{jt}^{\min}\, \sigma_{jt}^2\, L_{jt}^2}{V_{jt}}\right). \tag{32}$$

If tick size and minimum lot size have no influence on bid-ask spreads, then this relationship simplifies to $S_{jt}/P_{jt} \sim 1/L_{jt}$. It is exactly the relationship we have tested above for the Russian and U.S. equities markets. A promising direction for future research is to examine whether the R^2 in our regression can be improved by estimating an appropriate functional form for s.

Scaling Laws for Margins and Repo Haircuts.

Margin requirements determine the amount of collateral that traders deposit with exchanges or counterparties in order to protect them against potential losses due to adverse price movements or credit risk. Margin requirements should be sufficiently large to make losses from default negligible but not so large as to impede financial transactions.

Repurchase agreements (repo) are a form of over-collateralized borrowing in which a borrower sells a security to a lender with a commitment to buy it back in the future. The repo haircut is the amount by which the market value of a security exceeds the amount of cash that a borrower receives. Repo haircuts are similar to margins, because they also protect lenders from default risks.

Let H_{jt} denote the dollar margin or repo haircut, measured in dollars per share. Suppose that H_{jt} depends on the seven variables P_{jt}, V_{jt}, σ_{jt}^2, C, $K_{jt}^{\min}$, $Q_{jt}^{\min}$, and horizon T_{jt}. The parameter T_{jt} reflects the frequency of recalculating margin requirements or repo haircuts as well as the expected time to detect valuation problems and liquidate collateral. As before, dimensional analysis and leverage neutrality imply that the re-scaled percentage margin $H_{jt}\, L_{jt}/P_{jt}$ is a function h of the three re-scaled dimensionless and leverage-neutral variables $K_{jt}^{\min}$, $Q_{jt}^{\min}$, and T_{jt}:

$$\frac{H_{jt}}{P_{jt}} = \frac{1}{L_{jt}}\, h\left(\frac{K_{jt}^{\min}\, L_{jt}}{P_{jt}}, \frac{Q_{jt}^{\min}\, \sigma_{jt}^2\, L_{jt}^2}{V_{jt}}, \frac{T_{jt}\, \sigma_{jt}^2\, L_{jt}^2}{m^2}\right). \tag{33}$$

If minimum tick size, lot size, and collateral liquidation horizon are set optimally, then this relationship simplifies to $H_{jt}/P_{jt} \sim 1/L_{jt}$. The idea that H_{jt} is proportional to $1/L_{jt}$ captures the intuition that the optimal haircut depends not only on the standard deviation of returns σ_{jt} but also on the speed with which business time operates for the asset. Less liquid assets require larger haircuts than more liquid assets that are equally safe.

Scaling Laws for Trade Sizes and Number of Trades.

Our approach can be also used to derive more general scaling laws for the distribution of trade sizes and number of trades. Each bet of size Q_{jt} may be executed as a sequence of smaller trades. Let X_{jt} denote a trade, a fraction of a bet. Trades and bets have the same units but different underlying economics.

While it is reasonable to conjecture that the size of bets does not depend on minimum tick size or minimum lot size, the size of trades into which bets are "shredded" is usually affected by both of these frictions. For example, when tick size is restrictive, there are usually large quantities available for purchase or sale at best bids and offers; large bets thus may be executed all at once by cleaning out available bids and offers. It is also known that trades have become so small in recent years that minimum lot size is often a binding constraint, as shown by Kyle, Obizhaeva and Tuzun (2016) among others.

Suppose trade size depends on the six variables P_{jt}, V_{jt}, σ^2_{jt}, C, $K^{\min}_{jt}$, and $Q^{\min}_{jt}$. Since $P_{jt}\, X_{jt}/(C\, L_{jt})$ is dimensionless and leverage neutral, our algorithm leads to the following scaling laws for the probability distribution of trade sizes $\tilde{X}_{jt}$:

$$\text{Prob}\left\{\frac{P_{jt}\,\tilde{X}_{jt}}{C\,L_{jt}} < x\right\} = F^X_{jt}\left(x, \frac{K^{\min}_{jt}\,L_{jt}}{P_{jt}}, \frac{Q^{\min}_{jt}\,\sigma^2_{jt}\,L^2_{jt}}{V_{jt}}\right). \tag{34}$$

Similar scaling laws can be potentially obtained for distributions of bet sizes $\tilde{Q}_{jt}$, the quantities at the best bid and offer as well as for depth at tick levels throughout the limit order book.

Let N_{jt} denote the number of trades per day. Then the ratio N_{jt}/γ_{jt} denotes the average number of trades into which a bet is shredded. Suppose that the number of trades N_{jt} also depends on the six variables P_{jt}, V_{jt}, σ^2_{jt}, C, $K^{\min}_{jt}$, and $Q^{\min}_{jt}$. Following our algorithm, the number of trades N_{jt} satisfies

$$N_{jt} = \sigma^2_{jt}\,L^2_{jt}\,n\left(\frac{K^{\min}_{jt}\,L_{jt}}{P_{jt}}, \frac{Q^{\min}_{jt}\,\sigma^2_{jt}\,L^2_{jt}}{V_{jt}}\right). \tag{35}$$

If the function $n()$ is a constant, implying market frictions do not affect trading strategies of market participants, we obtain

$$N_{jt} = \text{const}\,\sigma^2_{jt}\,L^2_{jt}. \tag{36}$$

This is the equation tested earlier using Russian and U.S data. The more general

specification may generate more explanatory power for explaining how the number of trades varies across stocks.

Conclusion

There is a growing empirical evidence that the scaling laws discussed above match patterns in financial data, at least approximately. These scaling laws are found in data on transaction costs and order size distributions for institutional orders by Kyle and Obizhaeva (2016*c*); in data on trades executed in the U.S. and South Korean equities markets by Kyle, Obizhaeva and Tuzun (2016) and Bae et al. (2016); in Thomson Reuters data on news articles by Kyle et al. (2014); and in intraday trading patterns of the S&P E-mini futures market by Andersen et al. (2016).

The ideas discussed in this paper suggest new directions for empirical market microstructure research. Checking the validity of scaling laws in other samples, identifying boundaries of their applicability, improving the accuracy of estimates, determining specific functional forms for f, t^*, s, h^*, F^x_{jt}, n, and the triangulation of proportionality constants are important tasks for future research.

Our research here is relevant for risk managers and traders, who seek to minimize and measure market impact costs. It also establishes politically neutral, scientific benchmarks for numerous policy issues connected with market microstructure such as setting tick sizes and minimum lot sizes as well as position limits, margin requirements, and repo haircuts. As discussed in Kyle and Obizhaeva (2016*a*), such research is highly relevant for the economic analysis of market crashes like the U.S. stock market "flash crash" of May 2010 examined by the Staffs of the CFTC and SEC (2010b), the U.S. bond market "flash rally" of October 2014 examined in the Joint Staff Report (2015), as well as the ruble crash of December 2014 analyzed by Obizhaeva (2016). Lastly, it directly relates to designing liquidity management tools, one of the central issues addressed by recent regulatory initiatives.

REFERENCES

Andersen, Torben G., Oleg Bondarenko, Albert S. Kyle, and Anna A. Obizhaeva. 2016. "Intraday Trading Invariance in the E-mini S&P 500 Futures Market." Working Paper. Available at SSRN: https://ssrn.com/abstract=2693810.

Bae, Kyoung-hun, Albert S. Kyle, Eun Jung Lee, and Anna A. Obizhaeva. 2016. "Invariance of Buy-Sell Switching Points." Working Paper. Available at SSRN: https://ssrn.com/abstract=2730770.

Barenblatt, Grigory Isaakovich. 1996. *Scaling, Self-Similarity, and Intermediate Asymptotics.* Cambridge University Press.

Black, Fischer, and Myron Scholes. 1973. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, 81(3): 637–654.

Bouchaud, Jean-Philippe, Doyne J. Farmer, and Fabrizio Lillo. 2009. "How Markets Slowly Digest Changes in Supply and Demand." In *Handbook of Financial Markets: Dynamics and Evolution.* 47–160. North Holland.

Foucault, Thierry, Marco Pagano, and Ailsa Roell. 2013. *Market Liquidity: Theory, Evidence, and Policy.* Oxford University Press.

Gabaix, Xavier. 2009. "Power Laws in Economics and Finance." *Annual Review of Economics*, 121: 255–293.

Gabaix, Xavier, Parameswaran Gopikrishnan, Vasiliki Plerou, and Eugene H. Stanley. 2003. "A Theory of Power-Law Distributions in Financial Market Fluctuations." *Nature*, 423(6937): 267–270.

Gabaix, Xavier, Parameswaran Gopikrishnan, Vasiliki Plerou, and Eugene H. Stanley. 2006. "Institutional Investors and Stock Market Volatility." *Quarterly Journal of Economics*, 121(2): 461–504.

Grinold, Richard C., and Ronald N. Kahn. 1995. *Active Portfolio Management.* Probus Publishing.

Joint Staff Report. 2015. *Joint Staff Report: The U.S. Treasury Market on October 15, 2014.* Joint staff report by U.S. Department of the Treasury, Board of Governors of the Federal Reserve System, Federal Reserve Bank of New York, U.S. Securities and Exchange Commission, U.S. Commodity Futures Trading Commission. July 13, 2015.

Kolmogorov, Andrey Nikolaevich. 1941. "Local Structure of Turbulence in an Incompressible Fluid for Very Large Reynolds Numbers." *Dokl. Acad. Nauk SSSR*, 31(4): 299–303. Reprinted in Procedings of the Royal Society London A 434, 9-13, 1991.

Kyle, Albert S. 1985. "Continuous Auctions and Insider Trading." *Econometrica*, 53(6): 1315–1335.

Kyle, Albert S., and Anna A. Obizhaeva. 2016*a*. "Large Bets and Stock Market Crashes." Working Paper. Available at SSRN: https://ssrn.com/abstract=2023776.

Kyle, Albert S., and Anna A. Obizhaeva. 2016*b*. "Market Microstructure Invariance: A Dynamic Equilibrium Model." Working Paper. Available at SSRN: https://ssrn.com/abstract=2749531.

Kyle, Albert S., and Anna A. Obizhaeva. 2016*c*. "Market Microstructure Invariance: Empirical Hypotheses." *Econometrica*, 84(4): 1345–1404.

Kyle, Albert S., Anna A. Obizhaeva, and Tugkan Tuzun. 2016. "Microstructure Invariance in U.S. Stock Market Trades." FEDS Working Paper No. 2016-034. Available at SSRN: https://ssrn.com/abstract=2774039.

Kyle, Albert S., Anna A. Obizhaeva, and Yajun Wang. 2017. "Smooth Trading with Overconfidence and Market Power." *Review of Economic Studies*, Posted March 8: https://doi.org/10.1093/restud/rdx017.

Kyle, Albert S., Anna A. Obizhaeva, Nitish R. Sinha, and Tugkan Tuzun. 2014. "News Articles and the Invariance Hypothesis." Working Paper. Available at SSRN: https://ssrn.com/abstract=1786124.

Modigliani, Franco, and Merton H. Miller. 1958. "The Cost of Capital, Corporation Finance and the Theory of Investment." *American Economic Review*, 48(3): 261–297.

Newman, M.E.J. 2005. "Power Laws, Pareto Distributions and Zipf's Law." *Contemporary Physics*, 46(5): 323–351.

Obizhaeva, Anna A. 2016. "The Russian Ruble Crisis of December 2014." *Voprosy Ekonomiki*, 5: 66–86.

Pohl, Mathias, Alexander Ristig, Walter Schachermayer, and Ludovic Tangpi. 2017. "The amazing power of dimensional analysis: Quantifying market impact." Working Paper. Available at https://arxiv.org/abs/1702.05434.

Sonin, Ain A. 2001. *The Physical Basis of Dimensional Analysis.* Cambridge, MA: Department of Mechanical Engineering, MIT.

Sornette, Didier. 2004. *Critical Phenomena in Natural Sciences, Chaos, Fractals, Self-organization and Disorder: Concepts and Tools.* Springer, Heidelberg.

Staffs of the CFTC and SEC. 2010b. *Findings Regarding the Market Events of May 6, 2010.* Report of the Staffs of the CFTC and SEC to the Joint Advisory Committee on Emerging Regulatory Issues. September 30, 2010. Final Report. Available at https://www.sec.gov/news/studies/2010/marketevents-report.pdf.