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"Cash providers: asset dissemination over intermediation chain"

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Cash providers: asset dissemination over intermediation chains*

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Preliminary draft

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The recent crisis showed how financial assets originated at one place could get disseminated in the financial system and eventually appear on the balance sheet of distant financial institutions, not directly connected to the originating bank. We formulate a model to study the building up of intermediation chains involving OTC transactions in the shadow banking sector. We analyze the incentives for an institution to provide liquidity and serve as an intermediary by reselling the assets it buys. Higher liquidity in the system or lower haircuts on the asset make intermediation less necessary, but also increase the level of origination, which ultimately disseminates the asset among more institutions. This dissemination process endogenously generates common exposure, with implications for systemic risk.

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1 Introduction

Recent years have seen a growing importance of banks' reliance on wholesale funding, in particular through the development of "shadow banks" with no direct access to deposits that need to rely on other institutions to finance their investment opportunities. They do so through various techniques such as repackaging, reselling, or collateralized borrowing. Even for more traditional banks, the search for cheap funding is a key determinant of profitability (see e.g. Hanson, Kashyap, and Stein (2011)). As highlighted by the subprime crisis, this search for funds may result in long chains of intermediation, in which part of the assets originated in some U.S. states transited through various banks until reaching some European banks.

This paper offers a model of the building up of intermediation chains based on the search for liquidity. We analyze how the dissemination of assets across such chains and their length are affected by the architecture of the financial system, the supply of liquidity in the system, and some properties of the traded asset, in particular its collateral value.

The typical situation we have in mind is the following. An 'originator' has an opportunity to extend loans to households or firms but does not have enough cash to finance the operation. He can use various sources of financing: unsecured borrowing at the market rate, secured borrowing using the loans as collateral, sale of some of the loans to other institutions. In the last case, the originator will typically resort to an OTC transaction and make offers to some intermediaries with whom he usually trades. If an intermediary accepts the offer but does not have enough cash or is not willing to use it all to finance the purchase, he may, in turn, rely on the three channels described above. If he also chooses to sell some of the assets bought (possibly at a different price), the process goes on until nobody wants to disseminate assets any further. In this model the only gains from trade come from differences in funding costs and in available cash across different agents, other elements like diversification or differences in information are on purpose kept out of the model.

Each intermediary in the chain makes an offer to his partners that solves a trade-off between getting more funding via selling more assets or at a higher price, and making an offer that is attractive enough to have a high chance of being accepted. This trade-off depends on the financing needs of the intermediary, as higher needs will make the intermediary more eager to ensure that his offer will be accepted. It also depends on the funding costs of the intermediaries who receive the offer, which themselves depend on how much they can get from other intermediaries, and so on. We show that in equilibrium all intermediaries along an intermediation chain finance their purchases first by using all their cash, and then always sell part of their assets and use the rest as collateral for secured borrowing.

We can then derive a number of implications. A first variable of interest is the price at which the asset is sold by the different intermediaries. We show that intermediaries concede larger rebates when their financing needs are higher, which implies that the negotiated price goes up along the intermediation chain. Moreover, the rebate is proportional to the haircut on the asset when it is used as collateral. Intermediaries can more easily finance the purchase of the asset when it has a lower haircut, which makes its price closer to its fundamental value.

We then study the determinants of the intermediation chain's length, and show that it results from the interaction of several effects. An increase in the supply of liquidity in the system (through various parameters such as for instance lower haircuts or more intermediaries) makes it optimal to use shorter intermediation chains to finance a given volume of originated loans. As funding becomes cheaper however, the originated volume increases, so that the total impact is ambiguous. If origination is very sensitive to funding costs, an increase in liquidity eventually leads to longer chains and more dissemination of the asset.

This has interesting implications if the collateral value is too high because tail risk is underestimated, in which case a high asset origination and dissemination will follow simply due to financial intermediaries' search for cheap financing. The haircut in the model can also be interpreted as set by the central bank in its liquidity provision operations, in which case the model gives a framework to understand how decisions to accept new assets as collateral can affect the market for these assets and investment.

This formation of intermediation chains also has important consequences *ex post* when the asset return is realized. Since all intermediaries along a chain provide their cash entirely to avoid costly unsecured borrowing, after the transactions take place all intermediaries along a chain are 'tight' in cash and have engaged all their assets as collateral. If the asset has a lower return than expected, specifically lower than the collateral value, they all at the same time do not have enough cash to repay their creditors, except maybe the last intermediary in the chain. This 'systemic' event is not due to contagion, since our intermediaries have no engagement between themselves, but to a common exposure and the fact that strategic trades generate an extreme use of their cash. Exploiting the joint determination of the network of realized trades and of the asset quantities held by the different banks thus gives new results on systemic risk, as the number of distressed intermediaries is directly related to the number of intermediaries in the chain.

The end of this section relates our work to the literature. Then Section 2 describes the model, Section 3 solves for the equilibrium, Sections 4 and 4.2 derive the implications of the model for OTC trading and for systemic risk, respectively. Section 5 concludes.

Related literature. In terms of results, this work is related to recent strands in the literature on securitization and on shadow banking. The literature on securitization is surveyed in Gorton and Metrick (2011). Our contribution to the understanding of securitization activity is to show how the functioning of repo markets and OTC markets for the securitized products simultaneously determine the price at which these products can be resold, and thus the quantity originated. Erel, Nadauld, and Stulz (2012) show that the holdings of highly-rated securitization tranches differed widely across U.S. banks before the crisis and that differences are well explained by securitization activity. They do not look however at non-bank buyers of these tranches, through whom the assets can be disseminated outside the banking system.

Adrian and Ashcraft (2012) survey the literature on shadow banking. This literature has looked at the creation of credit chains, in particular through repo markets. Krishnamurthy, Nagel, and Orlov (2012) in particular show that the contraction of repo markets affected more key dealer banks. We offer a view complementary to this literature by looking at the impact of these markets on the dissemination of assets. A related paper is Gennaioli, Shleifer, and Vishny (2013), where securitization and the sale of securitized assets are important financing means, but their framework does not allow for the building of intermediation chains.

This paper is also connected to recent theories of freezes on OTC markets, see for instance Acharya, Gale, and Yorulmazer (2011), Caballero and Simsek (2013) or Camargo and Lester (2011), based on informational problems. In contrast, in our model drops in OTC volumes can be triggered by changes in haircuts or in the topology of the OTC network. Another potential application of our model would be to dealer networks, see for instance Li and Schuerhoff (2012) or Neklyudov (2012).

Methodologically, our approach is a compromise between two strands of the literature. On the one hand, papers like Duffie, Garleanu, and Pedersen (2005), Duffie, Garleanu, and Pedersen (2007), Lagos and Rocheteau (2007) or Lagos, Rocheteau, and Weill (2009) study how search frictions affect the pricing and the liquidity of assets on OTC markets. The most related paper in this literature is Atkeson, Eisfeldt, and Weill (2012), which introduces profits from intermediation. It is difficult however to study intermediation chains and origination in these models, which rely on a stationary environment.

On the other hand, the network literature has extensively studied chains and contagion, for instance Gai and Kapadia (2010), and what optimal banking networks would look like, see Castiglionesi and Navarro (2007). Anand *et al.* (2012) in particular study contagion through common exposure, and not only through banks' balance sheets. It is difficult however to study analytically endogenous trading decisions and derive predictions on prices and volumes with this approach. Bluhm, Faia, and Krahnen (2012) for instance include endogenous prices in a quite general network setup, but their approach is different as the objective is to calibrate the model and obtain numerical results. Blume *et al.* (2009) or Gofman (2011) come closer to our objective but do not allow to study asset dissemination, as intermediaries cannot keep part of the assets they resell. Malamud and Rostek (2012) develop a very general model of decentralized trading and mainly study the implications on an asset's liquidity, but not the build-up of intermediation chains per se.

The present paper studies a question that is best set up in a network framework, but by assuming a particular network structure we can embed the game in a tractable stationary problem closer to the search approach, and use this stationarity to derive an analytical solution and qualitative implications of the model.

2 The model

2.1 Assumptions and definitions

The asset and the originator: each unit of an asset delivers a random revenue $\tilde{\rho} \in \mathbb{R}^+$ according to a continuous distribution $F(.)$. We denote ρ the expected value of $\tilde{\rho}$. An "originator" chooses to originate k assets at a cost $C(k)$. C satisfies $C(0) = 0$, is strictly increasing and convex ($C' > 0, C'' \geq 0$). The asset could for instance be loans to households or companies, in which case C is the sum of the amount lent plus search or monitoring costs.

To simplify, the originator does not have any cash and needs to finance $C(k)$ either by selling some units of this asset, using unsecured borrowing, or borrowing using the assets as collateral. There is no liquid market for the asset, which can only be sold over-the counter to an intermediary. This intermediary has some cash but maybe not enough to buy all the assets, in which case the difference must be financed using the same three options. The initial volume k will thus be distributed on a network of intermediaries selling sequentially on an over-the-counter market.

Sources of funding: each intermediary starts with some amount of cash ω independently drawn from a distribution $G(.)$ on $\mathbb{R}^+$. Cash is invested at the safe interest rate, normalized to zero. Additional sources of cash are secured and unsecured borrowing. Each unit of asset that is not sold can be used as collateral. Against one unit left as collateral, external financiers are ready to lend some amount ℓ at the safe interest rate, as long as the probability of the asset's value being lower than ℓ is lower than some small value α (as e.g. in Brunnermeier and Pedersen (2009)). Thus we need $\Pr(\tilde{\rho} < \ell) \leq \alpha$, and ℓ can be thought of as the value-at-risk of the asset at level α and will be considered a parameter of the model. $\rho - \ell$ is the *haircut* per unit of collateral.

To complement secured borrowing, each intermediary can also borrow on the unsecured market at an exogenous interest rate r, so that he has to repay $1 + r$ after $\tilde{\rho}$ is realized. No collateral is required.

All agents are supposed to be risk neutral and have unlimited liability. We can think of them as desks in large institutions, whose debt is backed by the institution so that the risk of unsecured lending is not linked to the desk's decisions (small compared to the total positions of the institution). Another interpretation of unsecured lending is that the desk is using the institution's capital, in which case r would be an internal rate of return.

The trading process: we assume that each agent knows d other intermediaries, among them each one has a probability q to be available for trade (independent draws). The tree of all intermediaries available for trade whom the originator could potentially reach is thus the outcome of a Galton-Watson process, frequently analyzed in the computer science literature (see e.g. Kleinberg and Raghavan (2005)).

An intermediary I at a given layer of the tree decides on a take-it or leave-it offer (p, v) made to all his partners in the following layer. As cash is invested at the safe interest rate, I can attract the liquidity of other intermediaries by offering them a positive return. Each intermediary who receives the offer decides whether to accept it, in which case he buys v units of the asset at price p, or refuse. The intermediary decides on an offer before he learns how many partners are available and we make the simplifying assumption that only one deal at most can be done. If several intermediaries accept the offer, then I randomly chooses one of them to trade with.

Finally, when making an offer, I only knows the distribution G but not the amount of cash owned other intermediaries. It is useful to define the probability $H(\omega)$ that at least one intermediary at a given layer of the network is active and has more cash than ω:

$$H(\omega) = 1 - [1 - q + qG(\omega)]^d \tag{1}$$

We assume that H is log-concave.[1] As a result, the function $H(\omega)\omega$ is first increasing and then decreasing to 0, and we denote ω^* the maximum of this function.

Figure 1 gives an example with the first two layers of a network starting with the originator. One intermediary is inactive in the first layer, one rejects the offer, and the last one accepts it. This intermediary makes a new offer, which is rejected by two intermediaries and accepted by one[2].

[1] The log-concavity of H is implied by that of the distribution G. Most common distributions (Gaussian, uniform, beta, or exponential) are log-concave.

[2] All figures and tables are in the Appendix A.

[Insert Fig. 1 here.]

Different assumptions could of course be made about the trading protocol. What matters for our results is that the OTC market we consider is not frictionless: the originator cannot simply borrow from cash-rich intermediaries at a competitive price, so that the network structure and the distribution of cash have an impact on the originator's funding costs. Intermediaries could use more sophisticated trading mechanisms, which would presumably make the situation closer to a centralized environment but without qualitatively altering the main trade-offs we study.

2.2 The financing decision

We describe here the problem faced by intermediary I, who has himself received an offer, denoted by (p_0, v_0)[3], and can make a new offer, thus acting as an intermediary. To decide whether or not to accept the initial offer, I evaluates how much profit can be achieved by accepting it, and accepts if profit is non-negative. This profit depends on how the purchase is financed. In particular, I's optimal behavior will depend on his that an offer to sell some units of the asset is accepted.

Financing needs: assume I has ω in cash and accepts the offer (p_0, v_0). Since the v_0 units can be used as collateral, the amount $\ell v_0 + \omega$ can be obtained at zero interest to finance the purchase $p_0 v_0$. I's *financing needs* are defined as:

$$y = max[(p_0 - \ell)v_0 - \omega, 0] \tag{2}$$

This is the quantity that I needs to borrow if he makes no offer, in which case his profit is

$$(\rho - p_0)v_0 - ry. \tag{3}$$

Offers: let I make an offer (p, v) to his usual partners. He cannot sell more units than he buys and v has to be lower than v_0.

[3]Once k assets have been originated, the originator is in the same position as an intermediary who bought k assets at price $C(k)/k$.

If (p, v) is not accepted, I is in the same situation as if he had made no offer: y must be borrowed on the unsecured market at the cost ry, and the expected profit is as in (3).

If (p, v) is accepted, I keeps $v_0 - v$ units, receives pv from the sales, and can borrow $\ell(v_0 - v)$ on the unsecured market. The amount $pv + \ell(v_0 - v) + \omega$ is thus available at zero interest cost. If this amount is lower than $p_0 v_0$, the remaining financing needs are equal to $p_0 v_0 - (pv + \ell(v_0 - v) + \omega)$, which can be rewritten as $y - (p - \ell)v$: the financing needs are diminished by the value of the sale minus the loss in collateral. In the sequel we call $(p - \ell)v$ the *net cash value* of the transaction. There are three cases to consider:

(i) the offer's net cash value *under finances* the needs: $y - (p - \ell)v$ is positive and has to be borrowed on the unsecured market; (ii) the offer's net cash value *over finances* the needs, $y-(p-\ell)v$ is negative, and the intermediary keeps some cash which earns zero interest (this is surely the case when y is null); (iii) the net cash value *exactly covers* the needs: no unsecured borrowing is needed and no cash is retained. In any case, the interests to be paid due to unsecured borrowing or those received due to extra cash are equal to $r \max[y - (p - \ell)v, 0]$. The profit expected from offer (p, v) if it is accepted is thus equal to:

$$\pi = \rho(v_0 - v) - p_0 v_0 + pv - r \max[y - (p - \ell)v, 0]. \tag{4}$$

Beliefs on acceptance: an offer may not be accepted because no partner is available or no available partner finds it attractive. An intermediary forms some belief $\Phi(p, v)$ that an offer (p, v) is accepted. The function Φ thus describes an intermediary's expectations about acceptance probabilities.

Profit, benefit and optimal behavior: given the belief described by Φ, using (3) and (4) and rearranging, the intermediary I's expected profit from offer (p, v) is

$$\pi(p_0, v_0, y; p, v) = (\rho - p_0)v_0 - ry + rB_\Phi(y; p, v) \tag{5}$$

$$\text{where } rB_\Phi(y; p, v) = \Phi(p, v) \times \begin{cases} ((1 + r)p - \rho - r\ell)v & \text{if } (p - \ell)v \leq y \\ (y - (\rho - p)v) & \text{if } (p - \ell)v \geq y \end{cases} \tag{6}$$

$rB_\Phi(y;p,v)$ is the expected benefit for I due to the possibility that offer (p,v) is accepted, which reduces the financing costs. This expected reduction depends on I's total financing needs y and is surely less than $H(0)y$. Upon the receipt of offer (p_0,v_0), intermediary I determines his behavior in two steps.

First, I looks for an offer maximizing the expected benefit (6). Making offers more attractive by decreasing the asked price lowers the benefit drawn from the transaction in case of success, but lowers the risk of refusal and costly unsecured borrowing. Formally, an *optimal offer* solves

$$\Pi(p_0,v_0,y) = \max_{p\leq\rho, v\leq v_0} \pi(p_0,v_0,y;p,v). \tag{7}$$

Second, I accepts the proposed offer if this yields a non-negative profit in (5). (p_0,v_0) is thus accepted if the value $\Pi(p_0,v_0,y)$ is non-negative. Observe that for $p_0>\rho$ profit is negative, so that such an offer is never accepted. From now on we will thus consider only prices not larger than ρ.

Acceptance threshold: we now show that the intermediaries who accept an offer are those with enough cash. Observe that an intermediary with no financing needs surely makes a non-negative profit by accepting an offer since prices are less than the expected return ρ, as can be seen from (5). Under our assumption of unbounded support, there are such intermediaries. Second, profit is decreasing in y and thus increasing in the intermediary's cash ω. This implies that there is a threshold level of ω above which an offer (p_0,v_0) is accepted and this threshold is surely lower than $(p_0-\ell)v_0$. This defines the *threshold function* W_Φ by:

$$W_\Phi(p_0,v_0) = \inf\{\omega \text{ such that } \Pi(p_0,v_0,max[(p_0-\ell)v_0-\omega,0])\geq 0\}. \tag{8}$$

A first particular case is $p_0=\rho$. An intermediary accepts an offer with price ρ only if his financing needs are null: for $p_0=\rho$, the profit derived from accepting is bounded above by $-ry+rH(0)y$ since the benefit is lower than $H(0)y$. This upper bound is negative for any positive y.

A second case to consider is $p_0=\frac{\rho+r\ell}{1+r}$, a price we denote τ. At this price an intermediary without cash can still buy the asset, collateralize it and finance the remaining needs using unsecured borrowing without making a loss. A purchase at price τ can only yield a positive profit (equation (3)). Such an offer is thus surely accepted and $W_\Phi(\tau, v_0) = 0$.

We have shown the following properties if W_Φ, for any Φ:

Property 1. *Let intermediary I face offer (p_0, v_0) and have financing needs y.*

1. *If $y = 0$ then I accepts the offer without making a new one.*
2. *If $p_0 = \rho$ then the intermediary accepts the offer only if $y = 0$.*
3. *$W_\Phi(p_0, v_0) \leq (p_0 - \ell)v_0$, with equality if and only if $p_0 = \rho$.*
4. *$W_\Phi(\tau, v_0) = 0$.*

These properties imply that any reasonable Φ will be such that $\Phi(\tau, v) = H(0)$ for any v and $\Phi(p, v) > 0$ for $p \leq \rho$. Also, the probability of acceptance is bounded by the probability that at least one intermediary is available, hence $\Phi(p, v) \leq H(0)$.

When $p_0 < \rho$ there are thus intermediaries with $y > 0$ who accept the offer (p_0, v_0). By making a new offer (p, v) with $p = \rho$ and $v \leq v_0$ the offer has some chance of being accepted and the benefit in case of acceptance is positive. This improves expected profit, which shows that an intermediary with $y > 0$ always makes an offer. Moreover, this offer is surely such that $p > \tau$: using (6), the benefit in case of acceptance is smaller than $-(\rho - p)v + r(p - \ell)v$, which is negative when $p \leq \tau$. These observations give us the following property:

Property 2. *An intermediary I facing offer (p_0, v_0) and with financing needs $y > 0$ always makes a new offer (p, v), with $\tau < p \leq \rho$ and $0 < v \leq v_0$.*

It follows from these properties that the game ends under two circumstances: no intermediary accepts the offer (sometimes because nobody receives it) or the selected receiver has no financing needs. In particular, the game surely ends if the offered price is ρ. On the contrary, the game goes on when an intermediary accepts the offer but needs extra financing. Surely offers are made at a price between τ and ρ.

3 Equilibrium

The game has two components: the investment decision of the originator, and the financing decisions of all intermediaries in the ensuing network. We first need to solve the second component, before analyzing the originator's choice.

3.1 The financing game

The situation faced by an intermediary is entirely characterized by the offer (p_0, v_0) it receives and his liquidity ω. The number of intermediaries between the initial proposer and the intermediary under consideration does not matter. The setting is thus Markovian with states described by (p_0, v_0, ω). We look for a *stationary equilibrium* of this financing game, such that two intermediaries in the same state behave identically. Equilibrium entails a condition of correct expectations on the behavior of the receivers: assuming that the intermediaries' probability of acceptance follows a certain function, the optimal response induces the same function.

Let an intermediary facing offer (p_0, v_0) expect receivers to accept an offer (p, v) with probability $\Phi(p, v)$. This intermediary will accept the offer (p_0, v_0) if his cash is above the threshold $W_\Phi(p_0, v_0)$ defined by (8). Hence the probability that (p_0, v_0) is accepted is $H(W_\Phi(p_0, v_0))$. Equilibrium requires that this is equal to $\Phi(p_0, v_0)$.

Definition 1. *An equilibrium is characterized by a threshold W and an acceptance probability Φ such that $W(p_0, v_0) = W_\Phi(p_0, v_0)$ and $\Phi(p_0, v_0) = H(W(p_0, v_0))$ for any (p_0, v_0).*

We construct an equilibrium of the financing game by considering a sequence of games. Define $\mathcal{G}_n$ as the financing game with the restriction that the game will stop after at most n offers are accepted. We start with $\mathcal{G}_1$; then, knowing the behavior of an intermediary facing a network with at most one layer of other intermediaries, we can iterate and consider the problem of an intermediary facing two layers, three layers and so on. We show that the optimal strategy in the financing game coincides with the optimal one in a game with a finite number of layers.

3.1.1 The game $\mathcal{G}_1$ with one layer only

Consider an intermediary I with financing needs y facing offer (p_0, v_0). I makes an offer to his partner intermediaries, who do not have access to any other intermediaries themselves. The game is solved by backward induction.

An intermediary R receiving offer (p, v) and with cash ω must finance $y_R = max[(p - \ell)v - \omega, 0]$ on the unsecured market. This yields the profit $(\rho - p)v - ry_R$. When $y_R > 0$, R optimally accepts the offer if $(\rho - p)v - r[(p - \ell)v - \omega]$ is non-negative. This yields the acceptance threshold

$$W_1(p, v) = \frac{p - \rho + r(p - \ell)}{r} v = \frac{1 + r}{r}(p - \tau)v.$$

It follows that the probability of acceptance is $H(W_1(p, v))$. We now consider I's optimal offer and show that it does not over finance I's financing needs. Assume by contradiction that $(p - \ell)v > y$. The expected benefit is $H(W_1(p, v)) \left(y - \frac{\rho - p}{r} v\right)$ by (6). As long as there is overfinancing, a small decrease in v thus increases the benefit in case of acceptance (or keeps it unchanged if $p = \rho$) and increases the probability that the offer is accepted. Such an offer cannot be optimal.

Thus (p, v) satisfies $(p - \ell)v \leq y$. Using (6), the benefit in case of acceptance is $\frac{1+r}{r}(p - \tau)v = W_1(p, v)$, which gives the expected benefit:

$$B_1(y_0; p, v) = H(W_1(p, v)) \times W_1(p, v).$$

The proposer's optimal strategy maximizes this expression over the (p, v) that satisfy $p \leq \rho, v \leq v_0$ and $(p - \ell)v \leq y$. Notice that choosing the optimal offer (p, v) is equivalent to choosing a certain target acceptance level $W_1(p, v)$. In $\mathcal{G}_1$ the benefit for I is exactly equal to the probability to find an intermediary with $\omega \geq W_1(p, v)$ times $W_1(p, v)$, as if I simply extracted the cash of the chosen target.

Remember that $\omega H(\omega)$ first increases and then decreases with a maximum reached at ω^*. An optimal strategy for the proposer is an offer (p, v) with $p = \rho$ and:
-$v = y/(\rho - \ell)$ if $y \leq \omega^*$, so that $W_1(p, v) = y$ and the expected benefit is $H(y)y$;
-$v = \omega^*/(\rho - \ell)$ if $y > \omega^*$, so that $W_1(p, v) = \omega^*$ and the expected benefit is $H(\omega^*)\omega^*$.

Since $y \leq (\rho - \ell)v_0$, in both cases we have $v \leq v_0$. Hence I's strategy is equivalent to choosing a target cash level that depends only on y (and not on v_0), the optimal target can be denoted $\Omega_1(y)$ and maximizes $\omega H(\omega)$ subject to $\omega \leq y$. In the case $y \leq \omega^*$, I sells just enough to cover his financing needs and does not need to borrow in case of acceptance. In the case $y > \omega^*$ however, I needs to borrow $y - \omega^*$ at the unsecured rate. The equilibrium benefit for I in $\mathcal{G}_1$ can thus be defined as:

$$B_1^*(y) = H(y)y, \text{ for } y \leq \omega^*, \; H(\omega^*)\omega^* \text{ for } y \geq \omega^*. \tag{9}$$

Denote $\overline{y}_1 = \omega^*$. The benefit $B_1^*(y)$ strictly increases for $y \leq \overline{y}_1$ and is constant for $y \geq \overline{y}_1$. Furthermore it does not depend on ρ, ℓ and r.

3.1.2 Equilibrium of the financing game

To construct an equilibrium of the financing game, we now iterate by adding new layers of intermediaries, creating the games $\mathcal{G}_n$. As we can see on Figure 3 where B_1^* is plotted in an example, when $y \geq \omega^*$ the benefit B_1^* does not increase any more because the proposer extracts the maximum expected cash from a single layer of intermediaries. With a second layer it is possible to reach a higher benefit up to a new threshold, which can be increased again by adding a third layer, and so on. For each game $\mathcal{G}_n$ we can define $B_n^*(y)$ the maximal expected benefit that an intermediary with financing needs y can make and $\Omega_n(y)$ the cash level targeted by an optimal offer.

We will show by induction that, for each y, there is a maximal value $N(y)$ for the number of rounds n beyond which the benefit $B_n^*(y)$ stops increasing and stays constant as more rounds are allowed and, furthermore, the optimal target $\Omega_n(y)$ does not change. We denote $B^*(y)$ and $\Omega(y)$ these limit values.

Example-single-valued distribution: before solving the general case it is instructive to start with a simple case where all intermediaries (almost surely) have the same level of cash ω^*. Clearly it is then optimal to target exactly this amount of cash, so that for any y we have $\Omega(y) = \omega^*$ and the probability that an offer is accepted is $H(\omega^*)$[4]. The offer itself however depends on the financing needs. Indeed, for $y > \omega^*$ the financing needs of the targeted intermediaries decrease by ω^* at each step. Thus, $N(y) + 1$ intermediaries are needed to finance y, where $N(y)$ is defined by $y = N(y)\omega^* + \omega, 0 < \omega < \omega^*$. We will show in the general

[4]The distribution G is degenerate in this example, but the notations extend easily by considering G as the limit of non-degenerate distributions.

case that $B^*(y) = H(\omega^*)(\omega^* + B^*(y - \omega^*))$, which by iteration implies here the following expression for the benefit:

$$B^*(y) = [H(\omega^*) + H(\omega^*)^2 + ... + H(\omega^*)^{N(y)}]\omega^* + H(\omega^*)^{N(y)+1}(y - N(y)\omega^*). \qquad (10)$$

This explains why the optimal offer depends on y even though its chance of being accepted is constant and equal to $H(\omega^*)$: an offering intermediary with $y > \omega^*$ takes into account that his direct partners will themselves make an offer which may be rejected, then his partners' partners will make offers and so on, until $(N(y) + 1)\omega^*$ covers the needs. To sum up, what matters is not only the direct acceptance probability but also the cumulative ones.

In the general case of a non degenerate distribution, there is an additional effect due to the fact that the targeted cash levels and the probability of acceptance vary with the level y. The functions Ω and B^* then have to satisfy the following:

Theorem 1. *The benefit B^* and the target Ω do not depend on ρ, ℓ, and r. They are characterized by:*

$$B^*(y) = \max_{\omega \leq y} H(\omega)(\omega + B^*(y - \omega)) \qquad (11)$$

$$\Omega(y) = arg \max_{\omega \leq y} H(\omega)(\omega + B^*(y - \omega)). \qquad (12)$$

B^* is increasing and bounded from above. For $y > 0$, $\Omega(y)$ is smaller than ω^*. Moreover, there exists $\underline{y}_1 \in (0, \omega^*)$ such that for $y \leq \underline{y}_1$ we have $B^*(y) = B_1^*(y)$ and $\Omega(y) = \Omega_1(y)$.

These functions characterize equilibrium strategies in the financing game: an intermediary I who accepts an offer and has positive financing needs y makes an offer $(P(y), V(y))$ that exactly covers his financing needs and gives zero profit to the target:

$$(P(y) - \ell)V(y) = y \text{ and } (\rho - P(y))V(y) - ry + r(\Omega(y) - B^*(y - \Omega(y))) = 0. \qquad (13)$$

The proof is given in the Appendix B. The main difficulty is to prove that an optimal offer always exactly covers its proposer's financing needs, which is not at all obvious: facing a general probability of acceptance $\Phi(p, v)$, it may be optimal to choose an offer that does not entirely cover an intermediary's need in order to guarantee a higher probability that it is accepted, or even to choose an offer that does more than cover financing needs. We observe that this is not the case in $\mathcal{G}_1$ however, and iterate in order to prove that the equilibrium Φ is such that choosing $(p-\ell)v = y$ is optimal. Figures 2 and 3 show plots of $\Omega(y)$ and $B^*(y)$ in an example, as well as $\Omega_n(y)$ and $B_n^*(y)$ for the first three iterations[5].

[Insert Fig. 2 and 3 here.]

3.1.3 Equilibrium policy and rewards

From Theorem 1, we know that the optimal offers are characterized by the equilibrium benefit and target functions. We examine here this offer policy, and derive some properties on the sequence of targeted intermediaries, that we call chain, and the intermediaries' profit.

Once the equilibrium functions B^* and Ω are solved for, we can define the financing needs of the targeted intermediary $Z(y)$:

$$Z(y) = y - \Omega(y). \tag{14}$$

We easily derive the following explicit formulas for the offered price and volume $P(y)$ and $V(y)$ from (13), as well as some of their properties.

Corollary 1.

$$P(y) = \frac{\rho y + r\ell(Z(y) - B^*(Z(y)))}{y + r(Z(y) - B^*(Z(y)))} \tag{15}$$

$$V(y) = \frac{y + r(Z(y) - B^*(Z(y)))}{\rho - \ell} \tag{16}$$

$Z(y)$ and $V(y)$, are non-decreasing in y. For $y \le \underline{y}_1$, $P(y) = \rho$ and $V(y) = y/(\rho-\ell)$. For y large enough, $P(y)$ converges to τ, while $V(y)$ is equivalent to $(1+r)y/(\rho-\ell)$.

Corollary 1 is fairly intuitive, the only point that is not straightforward is the monotony of Z, which is proven in the Appendix C.1. For high financing needs an intermediary chooses a low price, close to τ, as he is more eager to get financed. Since the price is lower he also

[5] The parameterization we use is introduced in Section 3.3.

has to sell more in order to cover more financing needs. As y goes down on the contrary the price increases and goes to ρ when we reach the region where $\Omega(y) = y$, that is $P(y) = \rho$.

An intermediary with financing needs y targets an intermediary with cash $\Omega(y)$ and financing needs $Z(y) = y - \Omega(y)$. This target will himself choose a target $\Omega(Z(y))$, and this target will have financing needs $Z(y) - \Omega(Z(y)) = Z^2(y)$. By iteration we can thus define a *targeted chain* starting with an intermediary with financing needs y. The length of this chain is $N(y)$, where $N(y)$ is the smallest integer j such that $Z^j(y) \leq \underline{y}_1$. As $\Omega(y) \leq \omega^*$ by Theorem 1, the financing needs of an intermediary along the targeted chain decrease by at most ω^* at each step. To reach financing needs of $\underline{y}_1$ starting with some y, one surely needs at least $\lfloor (y - \underline{y}_1)/\omega^* \rfloor + 1$ steps.

The same iteration can be made on the benefit function: we have $B^*(y) = H(\Omega(y))(\Omega(y) + B^*(Z(y)))$ and for $Z(y) > 0$, $B^*(Z(y)) = H(\Omega(Z(y)))(\Omega(Z(y)) + B^*(Z^2(y))$ and so on. We thus obtain:

Corollary 2. *The length of the targeted chain $N(y)$ is increasing in y and*

$$B^*(y) = H(\Omega(y)) \left(\Omega(y) + \sum_{i=1}^{N(y)} \Omega(Z^i(y)) \prod_{j=1}^{i} H(\Omega(Z^j(y))) \right). \tag{17}$$

This expression generalizes (10) which was obtained for a degenerate distribution. The expected benefit for the first intermediary as expressed in (17) is the sum of the cash used by each targeted intermediary in the chain, weighted by the probability of acceptance at each step. The decision of each intermediary can be understood as determining the targeted chain that maximizes the cash extracted along the chain. Notice that the total expected cash is higher than this quantity: at the first step for example, an accepting intermediary is likely to have more cash than the target, and the extra cash does not accrue to the originator. As a result, such an intermediary will make some profit.

Let us consider the intermediaries who receive proposal $(P(y), V(y))$. Those with cash $\omega \geq \Omega(y)$ accept the offer and their profit is

$$(\rho - P(y))V(y) - rz + rB^*(z) \text{ where } z = \max(y - \omega, 0). \tag{18}$$

As B^* is non-decreasing, we have:

Corollary 3. *The profit of an intermediary accepting* $(P(y), V(y))$ *increases in his cash* ω.

Each marginal unit of cash ω above $\Omega(y)$ brings a marginal profit of $r(1 - B^{*\prime}(y - \omega))$. This is positive for $\omega < y$, reflecting the benefit drawn from the sales of the asset to other intermediaries, but smaller than r, reflecting the fact that these intermediaries face a risk of refusal. When the financing needs become null the profit is constant and equal to $(\rho - P(y))V(y)$.

3.2 Origination and the full game

We now study the optimal origination decision. Remember that originating k units of the asset costs $C(k)$. As a benchmark, assume that the originator has no access to a network of partner intermediaries. The amount ℓk is borrowed at a null rate and, assuming no initial fund, $C(k) - \ell k$ needs to be borrowed at the unsecured rate r. The profit expected from the investment is thus

$$\rho k - C(k) - r(C(k) - \ell k).$$

As C is convex, a positive level of investment is profitable only if $C'(0)$ is low enough. More precisely, the optimal level of k is characterized by

$$\frac{\rho - C'(k)}{C'(k) - \ell} = r \text{ or } C'(k) = \tau \text{ if } C'(0) \leq \tau,\ k = 0 \text{ otherwise.} \tag{19}$$

With access to a network, the originator has access to the cash resources of other intermediaries and can finance the projects at a possibly lower cost. Once he has originated k assets, the originator is in the same situation as any other intermediary: he has k assets to sell or use as collateral, and financing needs $y(k) = C(k) - \ell k$. An optimal offer for financing needs k will provide the expected benefit $B^*(y(k))$, so that his profit is:

$$\Pi_O(k) = \rho k - C(k) - ry(k) + rB^*(y(k)). \tag{20}$$

As any other agent, the originator may choose $p = \rho$, in which case the game would always stop after the first round.

Proposition 1. *Let k be the optimal investment for the originator. It satisfies, if positive,*

$$\frac{\rho - C'(k)}{C'(k) - \ell} = r(1 - B^{*\prime}(y(k))) \text{ where } y(k) = C(k) - \ell k. \tag{21}$$

k is larger than its level without a network. k is increasing in ρ, and for ρ large enough, the originator makes an offer with $p < \rho$, and chains have more than two intermediaries with positive probability. k is non-increasing in r, and is non-decreasing in ℓ if B^* is concave.

See the Appendix C.2 for the proof. This implies that how many assets the originator sells to the next intermediary may increase or decrease when ρ is higher: a more valuable asset increases origination, but can decrease the incentives to sell. Similar remarks apply to the other financial parameters. Their impact on k, hence on $y(k)$ and the subsequent offers, are important to assess their overall impact on OTC trading.

Note that the impact of ℓ on origination depends on the concavity of B^*. B^* is concave in the case of a degenerate distribution[6]. We found no numerical example where this property did not hold in the general case but could not prove it formally. We thus mention explicitly when a result requires this assumption.

3.3 Numerical example

Before turning to the model's implications, we illustrate the equilibrium with a numerical example. We first choose a set of baseline parameters, chiefly for their computational convenience.[7] We solve for B^* and Ω and determine the optimal level of origination k. We then simulate the game 10.000 times and record empirical averages, at each level in the realized chain of intermediaries, of the volume v_0 faced by the intermediary, the price p and the volume v he offers, the volume $v_0 - v$ he keeps, his cash level ω, the cash level Ω he targets and his benefit B. We additionally record for each layer the probability p_a that an intermediary makes a new offer, p_r that nobody accepts an offer, and p_c that someone completes the volume and closes the chain. The averages of v_0, ω and $v_0 - v$ are weighted by the proportion

[6] It is piece-wise linear in y with kinks at points where $N(y)$ jumps to $N(y)+1$; at other points, the slope is equal to $H(\omega^*)^{N(y)+1}$.

[7] $\ell = 0.8$, $r = 1.2$, $\rho = 3.5$, $d = 3$, $q = 0.5$, $G(.)$ is the CDF of a Gamma distribution $\Gamma(\kappa, \theta)$ with $\kappa = 3, \theta = 0.1$. The cost function is $C(x) = x + (c/2) * x^2$, with $c = 1$.

of intermediaries making a new offer or completing the volume at a given layer, while p, v, Ω and B are weighted by the proportion of intermediaries making a new offer.

The results obtained with the baseline parameters are the following. The originator chooses to issue $k = 1.03$ units and targets a chain of size $N(y(k)) = 7$. On average, the realized chain has a size of 1.96 (the originator plus close to 2 intermediaries). The originator offers to sell 37% of the originated volume at a 23% discount. This offer is accepted by intermediaries who have at least $\omega = 0.13$ and has thus an acceptance probability of 81%, which yields a benefit B^* equal to 0.39 on average. Table 1 gives the results at each level.

[Insert Table 1 here.]

4 Implications for OTC trading and dissemination

We review in this section the implications of our model on asset dissemination, as well as empirical implications on trading behaviour on particular markets. To that purpose, we derive how a change in the environment as reflected by the different parameters affects the equilibrium outcome. We decompose the impact of each parameter into three effects: a *trading effect* is an impact on an intermediary's offer for given financing needs (a change of the function Ω), a *financing needs effect* is a change in the financing needs y for an intermediary receiving a given offer and having a given cash, and an *origination effect* is a change in the volume k chosen by the originator. These three effects allow us to disentangle the different channels through which each parameter affects observable variables such as originated volumes and chain size.

4.1 Implications on trading behaviour

We examine here the impact of the financial parameters ρ, ℓ and r on the offer of an intermediary with a given y. From the previous analysis, specifically Theorem 1, we know that an intermediary with given financing needs y makes an offer with a target $\Omega(y)$ that does not depend on the financial parameters. Thus the probability of acceptance is unaffected as well. However, the offer itself and the profits are affected. We have from (15) and (16):

$$\frac{\rho - P(y)}{\rho - \ell} = 1 - \frac{y}{y + r(Z(y) - B^*(Z(y)))} \tag{22}$$

$$V(y)(\rho - \ell) = y + r(Z(y) - B^*(Z(y))) \tag{23}$$

$\rho - P(y)$ is a *liquidity rebate*, which shows how much an intermediary is ready to concede in order to attract potential buyers. The terms on the right hand side of each equation are independent of ρ and ℓ. Recall that $\rho - \ell$ is interpreted as a haircut reflecting the difficulty to borrow against the asset as collateral. Direct inspection delivers the following implication on the impact of the financial parameters on the offer policy and profit.

Implication 1. *For given financing needs y, the target and the probability that the optimal offer is accepted are independent of ρ, ℓ and r. As for the offer itself:*

1. $P(y)$ is non-increasing in r and non-decreasing in ρ and ℓ. $V(y)$ is non-decreasing in r and ℓ and non-increasing in ρ.

2. The liquidity rebate is directly proportional and the offered volume inversely proportional to the haircut.

From 2, increasing ρ and ℓ by the same amount, thus not affecting the haircut $\rho - \ell$, leavex the liquidity rebate and the offered volume constant. An increase in the haircut makes it less attractive to keep the asset and use it as collateral, thus increasing the incentives to sell. But the same change makes it more difficult to finance the purchase of the asset, so that the seller has to concede a larger liquidity rebate, which turns out to be exactly proportional to the haircut.

An intermediary's offer depends on r even though his targeted intermediary does not. For this reason, the profit made by any intermediary with a cash level above the target will depend on r. To study this, denote $(P_r(y), V_r(y))$ the offer made by an intemediary with financing needs y. As the target is left unchanged, the zero profit condition yields $(\rho - P_r(y))V_r(y) = r(y - \Omega(y) - B^*(y - \Omega(y))$. Thus, the offer is adjusted to a change in r so that its net value, $(\rho - P_r(y))V_r(y)$, is proportional to r. As a result, an intermediary R with more cash than the target but not enough to finance the whole volume, i.e. $\Omega(y) < \omega < y$

$((P_r(y) - \ell)V_r(y) = y)$, has a profit of:

$$\begin{aligned} \Pi_R(\omega) &= (\rho - P_r(y))V_r(y) - r(y - \omega) + rB^*(y - \omega) \\ &= r[\omega - \Omega(y) - B^*(y - \Omega(y) + B^*(y - \omega)] \end{aligned}$$

For institutions with cash larger than y, the profit is equal to $\Pi_R(y)$. Thus Π_R is proportional to r. In particular, the increase in the offer's net value outweighs the additional financing cost due to the increase in r. Considering the impact of an increase in r on the whole chain, starting with a fixed investment k, we thus obtain that the originator's profit decreases while that of all intermediaries either stay null or increase, as long as r is small enough so that the investment is profitable for the originator.

Implication 2. *Intermediation profit is proportional to the unsecured interest rate r, while the originator's profit is decreasing in r.*

Note that aggregate profit is decreasing in r because the use of unsecured borrowing occurs exactly under the same circumstances but a larger cost. Nonetheless, the profit of the intermediaries increase (as long as the investment is made). Ultimately cash allows to avoid using unsecured borrowing; the larger its associated cost, i.e. the larger r, the more cash is rewarded.

These implications can apply to different settings:

If secured borrowing in the model is interpreted as borrowing from other market participants at a competitively priced haircut, then changes in haircuts can be driven by market expectations. Simsek (2013) shows in a model of belief disagreement how optimism about upside returns may have led to both low haircuts and high asset prices (or low liquidity rebates) in the run-up to the crisis, which would also imply a high origination and long intermediation chains in our setup. In other periods disagreement about downside risks may have led to high haircuts, in which case we typically expect less origination in our model[8].

[8] In our model haircuts are determined by a competitive and exogenous supply of funding against collateral, whereas in Simsek (2013) investors endogenously choose to lend or borrow, so that haircuts and asset prices are simultaneously determined.

This view of a chain of cash-constrained intermediaries demanding liquidity to other intermediaries fits well the networks of dealers in municipal bonds studied by Green, Hollifield, and Schuerhoff (2007). Our intermediaries could be interpreted as dealers, and the ω of each intermediary as depending on a flow of buy orders coming from customers. It is less costly for a dealer facing strong demand from customers to buy the asset, while a dealer facing a lower demand will have to either hold the asset or sell it to a different dealer. The authors show that dealers demand liquidity on average, intermediation costs increase with trade size and the asset's risk, and decrease with transaction frequency. These observations are consistent with our theoretical results: $y - B^*(y)$ which is a measure of financing cost increases in y and thus with trade size in our model, the relation between costs and the asset's risk and thus its haircut is consistent with Implication 1. Finally, Li and Schuerhoff (2012) show that on the same market transaction prices go up on average along an intermediation chain: as dealers down the chain have lower financing needs, they make sale offers using higher prices and the liquidity rebate, $\rho - P(y)$, goes to zero when financing needs become small enough, which is consistent with Corollary 1.

Secured borrowing could also be interpreted as borrowing at the central bank against collateral, and the haircut is thus a choice variable for the central bank. Since the outbreak of the financial crisis, several central banks have accepted more assets as collateral when lending to banks, which can be seen as a change in haircut from 100% to some lower number. Our model could be tested by looking at how decisions taken on which assets are eligible or on the required haircuts have an impact on the volume of assets traded and the level of origination. Of particular interest in our model is the fact that lowering haircuts may encourage banks to transact more with each other, so that central bank liquidity is less needed and the total risk exposure of the central bank decreases. The optimal collateral policy of a central bank in this model is an interesting topic left for future research.

4.2 Chain size and the number of impacted layers

One of the distinctive features of our model is that it allows us to study the dissemination of assets across various layers of intermediaries, for instance different geographical regions or parts of the banking system. An intuitive measure of dissemination is to look at the size of the intermediation chain, which gives the number of regions in which some banks hold the asset. The impact of each parameter on this measure will be the result of its effects on trading strategies, financing needs, and origination.

Impact of ρ and r. It follows from the characterization of the optimal investment (Proposition 1) that the originated volume is non-decreasing in ρ and non-increasing in r. Since the financing needs and the trading strategies are unaffected by ρ or r, an increase in k directly translates into an increase in $y(k)$. Hence the size of the chain and the targeted levels in the chain all increase. In addition, as the originator's benefit function does not depend on ρ and k, the impact of these parameters on the originator's profit follows from the envelope theorem. This gives us the following implication:

Implication 3. *The investment of the originator and the size of the targeted chain are non-decreasing in the expected return ρ and non-increasing in the unsecured rate r.*

The originator's profit is increasing in the expected return ρ and in ℓ and decreasing in the unsecured rate r.

Assets that are expected to be more valuable will be originated in larger quantities, making the chain size higher. An increase in r on the contrary increases the originator's costs and thus reduces origination and chain size. Implication 3 is valid for any cost function. When k is fixed, the financing needs of the originator are unaffected by r and ρ. In that case the targeted chain is unchanged, and it follows directly from implication 1 that the sales of the originator decrease and the price increases in ρ.

Impact of ℓ. An increase in ℓ has a non-ambiguous positive effect on the origination level k when B^* is concave, so that the origination effect is positive. An increase in ℓ simultaneously reduces the financing needs $y(k)$, leading to shorter chains. Whether the origination or the financing needs effect dominates depends on the flexibility of origination. We consider two extreme cases:

Implication 4. *For a fixed k, the size of the targeted chain is decreasing in ℓ.*

Assume C to be linear: $C(k) = ck$, with $c > \tau$. Then the originated volume k, the originator's financing needs $y(k)$, the size of the targeted chain and the holdings of the originator and all intermediaries in a targeted chain all increase in the collateral value ℓ.

See the Appendix C.3 for the proof. In the linear case origination is quite flexible and the origination effect dominates, so that the originator's financing needs increase. The targeted intermediaries then have larger financing needs and all optimally target longer chains of intermediaries.

To summarize, unless the cost function is too convex, an increase in the collateral value of the asset will lead to more origination and higher total financing needs, so that more layers will be needed to cover them entirely. Optimism leading to higher collateral values in particular can be expected to favour the dissemination of an asset to many layers.

Impact of the network. d and q have no impact on an intermediary's financing needs, and it is straightforward to show that an increase in either parameter reduces the originator's costs and thus has a positive origination effect. But d and q also have a trading effect: if an intermediary has access to more trading partners or knows they have a higher probability of being active then this changes the probability $H(\omega)$ that an offer targeting type ω is accepted, which thus changes the strategy of an intermediary with given financing needs. The optimal offer ω must satisfy the first-order condition associated to (11), which can be written as:

$$\frac{H'(\omega)}{H(\omega)} + \frac{1 - B^{*\prime}(y-\omega)}{\omega + B^*(y-\omega)} \tag{24}$$

The parameters d and q have an effect on both members of this equation. First, there is a direct effect on the probability of acceptance and how this probability reacts to a change in the targeted ω. Second, the probability of acceptance is affected for all the subsequent intermediaries in the chain so that the shape of the benefit function is also altered.

The combination of both effects is in general ambiguous. However, we can study how d and q affect a lower bound on the length of the targeted chain. Remember that ω^* is the ω maximizing the function $\omega H(\omega)$, and thus depends only on parameters related to the network, d and q, and the distribution of cash $G(.)$. Moreover, ω^* satisfies by definition:

$$\frac{\omega^* H'(\omega^*)}{H(\omega^*)} = -1 \tag{25}$$

As is shown in the Appendix C.4, H'/H is always positively impacted by an increase in d or q, so that ω^* is increasing in both parameters. As $\Omega(y) \leq \omega^*$ by Theorem 1, the financing needs of an intermediary along the targeted chain decrease by at most ω^* at each step. To reach financing needs of $\underline{y}_1$ starting with some y, one surely needs at least $\lfloor (y - \underline{y}_1)/\omega^* \rfloor + 1$ steps. As moreover $\underline{y}_1 \leq \omega^*$, we obtain:

$$N(y) \geq \lfloor \frac{y}{\omega^*} \rfloor. \tag{26}$$

As ω^* increases in d and q, we deduce:

Implication 5. *The lower bound on the size of any targeted chain $N(y)$ decreases when d or q increase.*

See the Appendix C.4. This result illustrates well the importance of endogenizing the intermediaries' trading decisions. One may have expected that increasing the number of intermediaries at each layer or their probability to answer an offer would increase dissemination, as there is a higher probability to find someone ready to buy some of the asset. But precisely for this reason, intermediaries tend to endogenously choose offers that are accepted by fewer intermediaries, so that the length of the chain can ultimately decrease. This can be best understood by considering the extreme case $d \to +\infty$: for any financing need y an intermediary is sure that an offer to sell all his assets at price ρ will be accepted, so that it is always optimal to target $\omega = y$ and the chain has a length of 1. Observe however that the origination effect goes in the other direction: as d and q increase, it becomes cheaper to finance the origination of assets, so that $y(k)$ increases and thus $N(y(k))$ as well.

Impact of the distribution of cash G. The distribution $G(.)$ has an impact through the trading effect and the origination effect. For a degenerate distribution where all intermediaries almost surely have the same level of cash ω^*, we have seen that $\Omega(y) = \omega^*$ for any y, hence the size of the targeted chain is *equal* to the lower bound given in equation (26). This observation yields the following:

Implication 6. *If cash is distributed according to some cdf.* $G(.)$ *and* ω^* *maximizes* $\omega H(\omega)$, *the length of the chain targeted by an intermediary with financing needs* y *is larger than if all intermediaries had a cash level* ω^* *for sure, while the originated volume is lower.*

The asymmetric information between the proposer and the receiver thus has two effects on chain length: the proposer faces a risk that the next intermediaries have a low level of cash, in which case the offer may be rejected. This increases costs and thus reduces the originated volume. But the proposer also lowers the targeted cash level below ω^*, which means that intermediaries with lower financing needs can accept the offer, and more additional steps are needed before enough cash is extracted to cover the initial financing needs.

The analysis of this section can be summed up in the following table:

[Insert Table 2 here.]

Interestingly, each parameter in the model plays a role via at most two of the three possible effects we have identified. While all parameters have an impact on origination, d, q and the distribution also impact the size of the targeted chain by modifying trading decisions, and ℓ has an impact on financing needs but not on trading. While in all cases where there are two effects they go in opposite directions, the relative strength of each depends on the convexity of C, a more convex C implying a weaker origination effect.

Finally, the realized intermediation chain will in general be different from the targeted chain. We use simulations to study how changing several parameters from their baseline values affects the average length of the realized chain, the length of the targeted chain and the level of origination (top panels of Fig. 4, 5 and 6). We use both the baseline cost function and an infinitely convex cost function, thus suppressing the origination effect. We first look at the impact of the distribution $G(.)$ by changing its mean, and then its standard deviation. The impact of the mean is non-monotonic, even after shutting down the origination effect, which means that the trading effect can be ambiguous. A higher standard deviation seems to increase the size of the targeted chains (intermediaries are more cautious when making an offer) but decreases the probability that a given offer is accepted, leading to a negative impact on the realized length. ℓ tends to decrease chain length in this example (the financing needs effect dominates), so that lower haircuts imply less dissemination, even though the total originated volume increases.

[Insert Fig. 4, 5 and 6 here.]

4.3 Common exposure and dissemination

Consider an intermediary I who bought v_0 units of the asset and then sold $v < v_0$. I sold exactly enough assets to cover his financing needs, did not keep any cash and thus ended up with $v_0 - v$ units of the asset and pledged them as collateral against a debt of $\ell(v_0 - v)$. The same is true at each layer in the realized intermediation chain, except the last one. The asset's final value is random and equal to $\tilde{\rho}$. The value ℓ is supposed to be such that the probability to have $\tilde{\rho} < \ell$ is negligible and debt is essentially risk-free (as we have assumed so far). If however ℓ was chosen too optimistically and such a bad shock realizes, the value of each intermediary's assets is lower ex post than what he has to pay back to creditors:

Implication 7. *Whenever $\tilde{\rho} < \ell$, all intermediaries in the chain simultaneously lack of cash to reimburse their debt.*

A typical example of an abnormally high ℓ would be the run-up to the subprime crisis, when the quality of mortgage loans was severely overestimated (see e.g. Rajan, Seru, and Vig (2013)). In our model, when such an underestimation occurs origination increases, the asset is disseminated to more regions (layers), and an unanticipated negative shock simultaneously affects all participating intermediaries. Gennaioli, Shleifer, and Vishny (2013) also elaborate on this assumption of optimism in a model of shadow banking. Such asset dissemination through the shadow banking system seems to have been an important mechanism through which problems with securitised products based on U.S. subprime mortgages spread to other constituencies, including Europe, thereby creating an important international systemic risk.

Following Acharya, Pedersen, Philippon, and Richardson (2012), we can define a systemic event as happening when the total value of assets in the system falls below z times their volume for some positive target level z, and define SES_i the *systemic expected shortfall* of intermediary I as the expected difference between z times his asset holdings and his equity value, conditional on a systemic event. SES_I measures the expected contribution of I to the global shortfall in case of a systemic event. In this model it is equal to

$$SES_I = (v_0 - v)E(z - (\tilde{\rho} - \ell)|\tilde{\rho} - \ell < z)\Pr(\tilde{\rho} - \ell < z)$$

SES_I is proportional to $v_0 - v_I$ which are the *asset holdings* of intermediary I. What matters for risk is then the asset holdings of an intermediary at a given layer, which would correspond for instance to the number of units held by banks in a given geographical area, or pledged as collateral to other banks in the same region. As we have shown in Implication 4, following an increase in ℓ all intermediaries will choose to sell more of the asset. But this implies that they will also all receive more units, so that the total impact on the asset holdings is ambiguous. The bottom panels of Fig. 4, 5 and 6 show the impact of the collateral value and the distribution of cash on asset holdings in the first three levels of the chain. For a fixed origination a higher collateral value implies that more assets stay on the balance sheet of the first bank, and the asset holdings at the next levels decrease. When the origination effect is taken into account however, asset holdings increase at all levels, so that an underestimation of risk leads to more dissemination.

5 Conclusion

Securing a cheap access to funding has been an increasingly important determinant of profits for financial firms. When cash is in scarce supply, reducing funding costs is a powerful motive to trade besides more traditional ones such as diversification. We propose a model in which financial intermediaries differ only in their access to liquidity and show how these intermediaries' positions in an OTC market for a risky asset are determined by their liquidity needs and the ease with which they can sell to partner intermediaries. Intermediation chains arise naturally from such a model and assets get disseminated among many intermediaries, depending on how cash is distributed among them.

We show how funding conditions on the market for collateralized lending affect incentives to keep the assets and use them as collateral vs. selling them, and thus the liquidity premium at which these assets trade and the length of the intermediation chains. The volume of assets originated increases when the collateral value of the asset increases, which can happen for instance when agents are more optimistic about tail risk. Even though cheaper collateralized funding decreases the need to sell assets to other intermediaries, through this origination effect the length of intermediation chains and the dissemination of assets can increase.

As financial intermediaries are looking for cheaper sources of financing and cash is always their cheapest option, it always gets used first, and then all their assets are pledged as collateral. Haircuts are supposedly set so that the assets' liquidation value is larger than the value of the loan with a very high probability. When the asset's value turns out to be lower, then all members of the intermediation chain make losses at the same time due to their endogenously chosen common exposure. This implies in particular that cheaper financing through collateralized loans can increase the common exposure component of systemic risk.

Other applications such as the formation of rehypothecation chains or implications for a central bank's collateral policy are left for future research.

A Figures and Tables

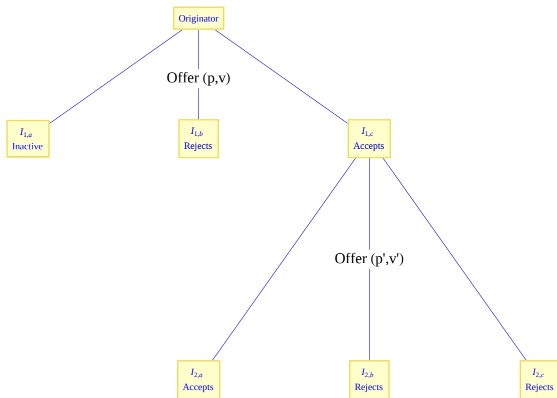

Figure 1: Two first layers of a network, example.

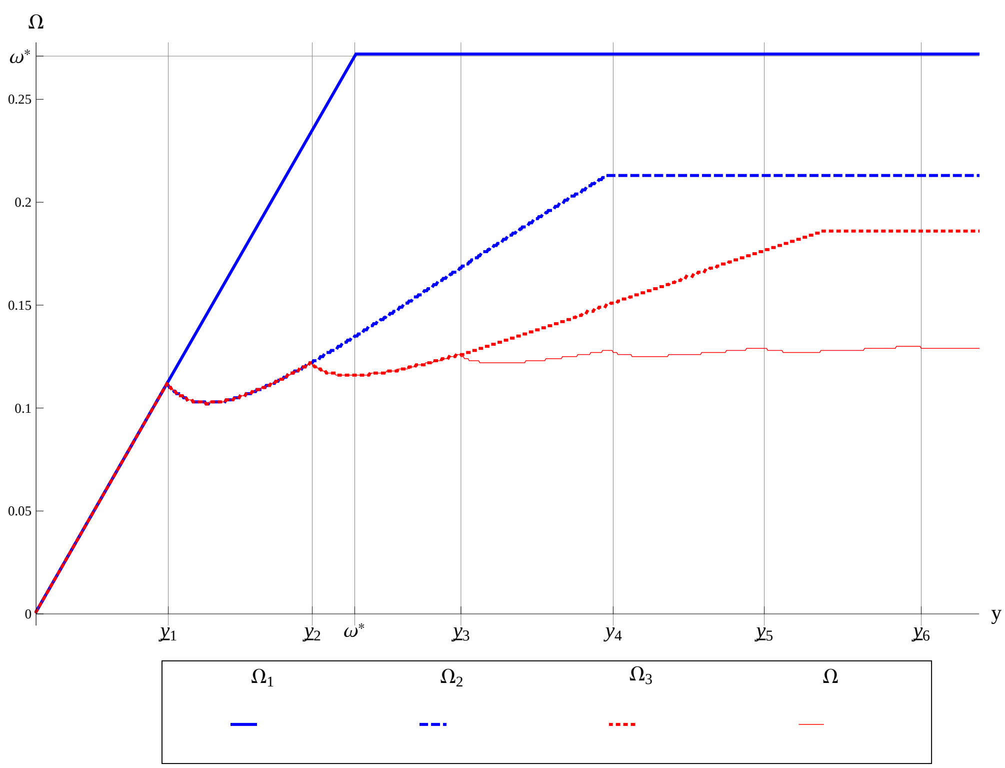

Figure 2: Equilibrium Ω and Ω_n for $n \leq 3$.

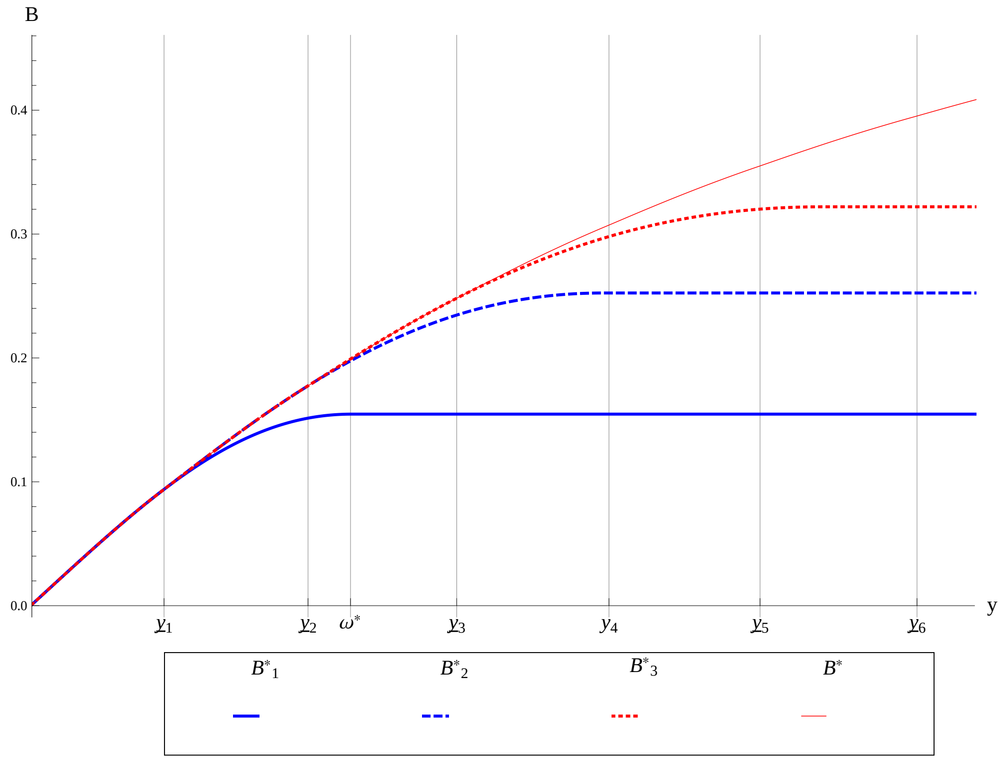

Figure 3: Equilibrium B^* and B_n^* for $n \leq 3$.

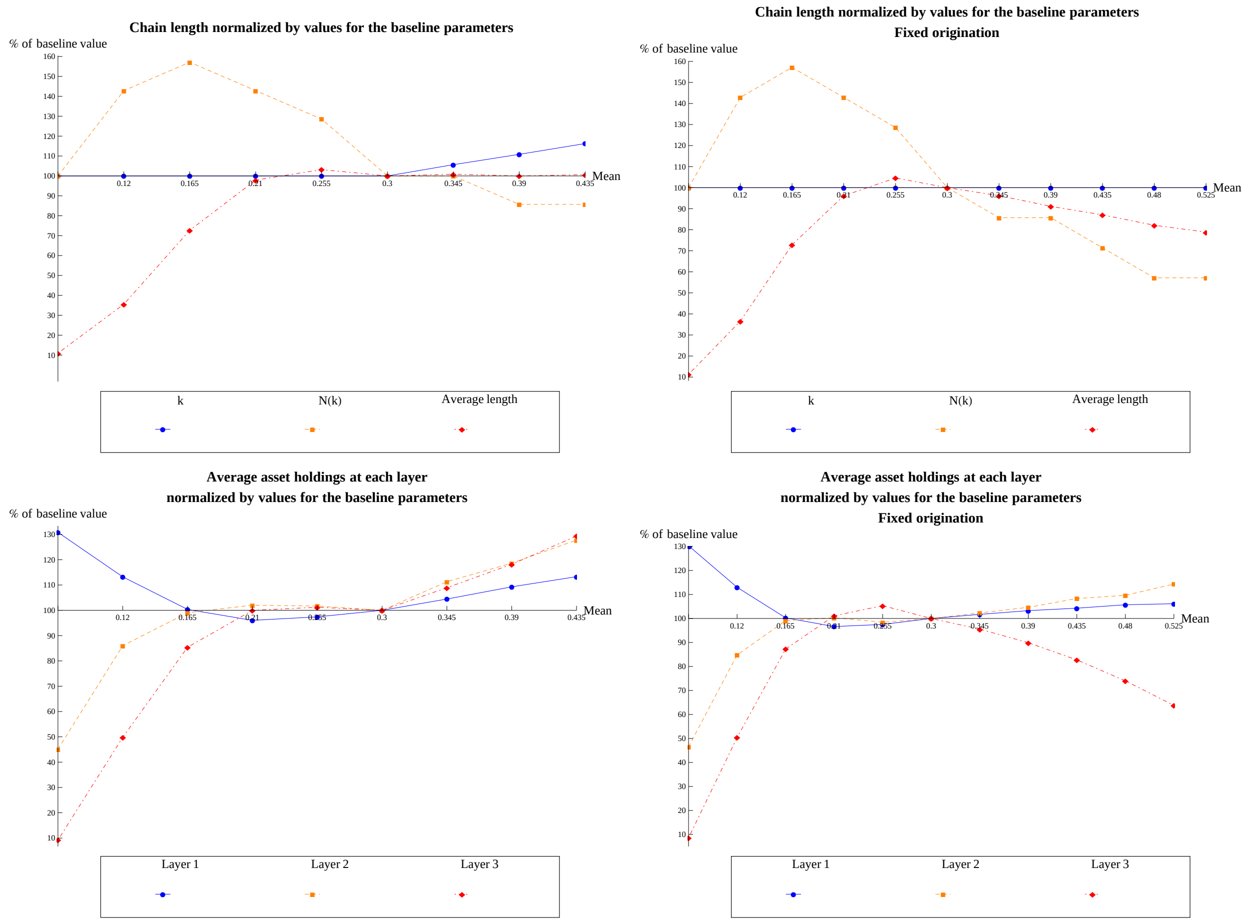

Figure 4: Impact of the distribution's mean on chain length and asset holdings.

Reading (e.g. top left panel): when the distribution's mean is 0.165 instead of its baseline value of 0.3, k, $N(k)$ and the averaged length of the realized chain are to about 100%, 155% and 75% of their values under the baseline parameters, respectively.

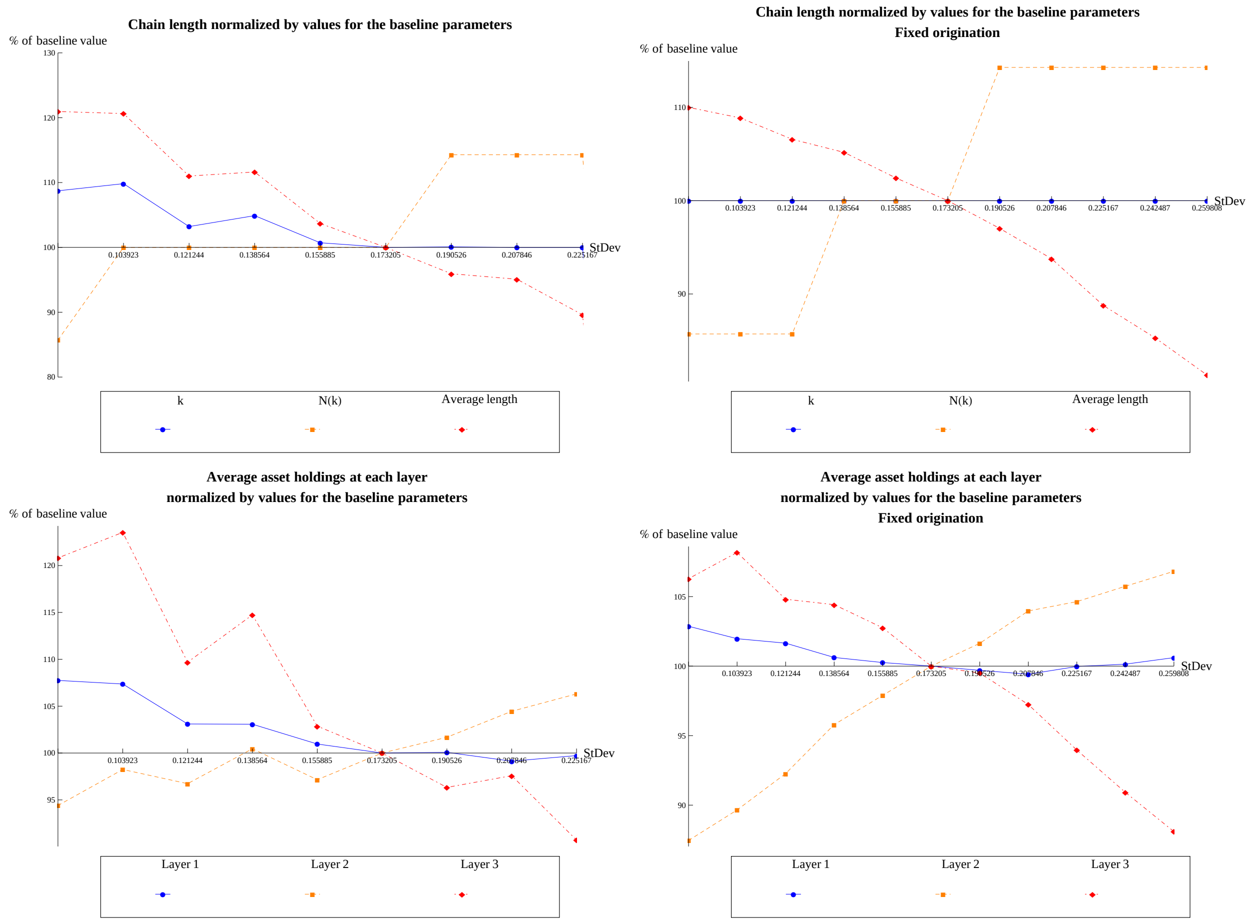

Figure 5: Impact of the distribution's standard deviation on chain length and asset holdings.

Reading (e.g. top left panel): when the distribution's standard deviation is 0.12 instead of its baseline value of 0.17, k, $N(k)$ and the averaged length of the realized chain are to about 103%, 100% and 110% of their values under the baseline parameters, respectively.

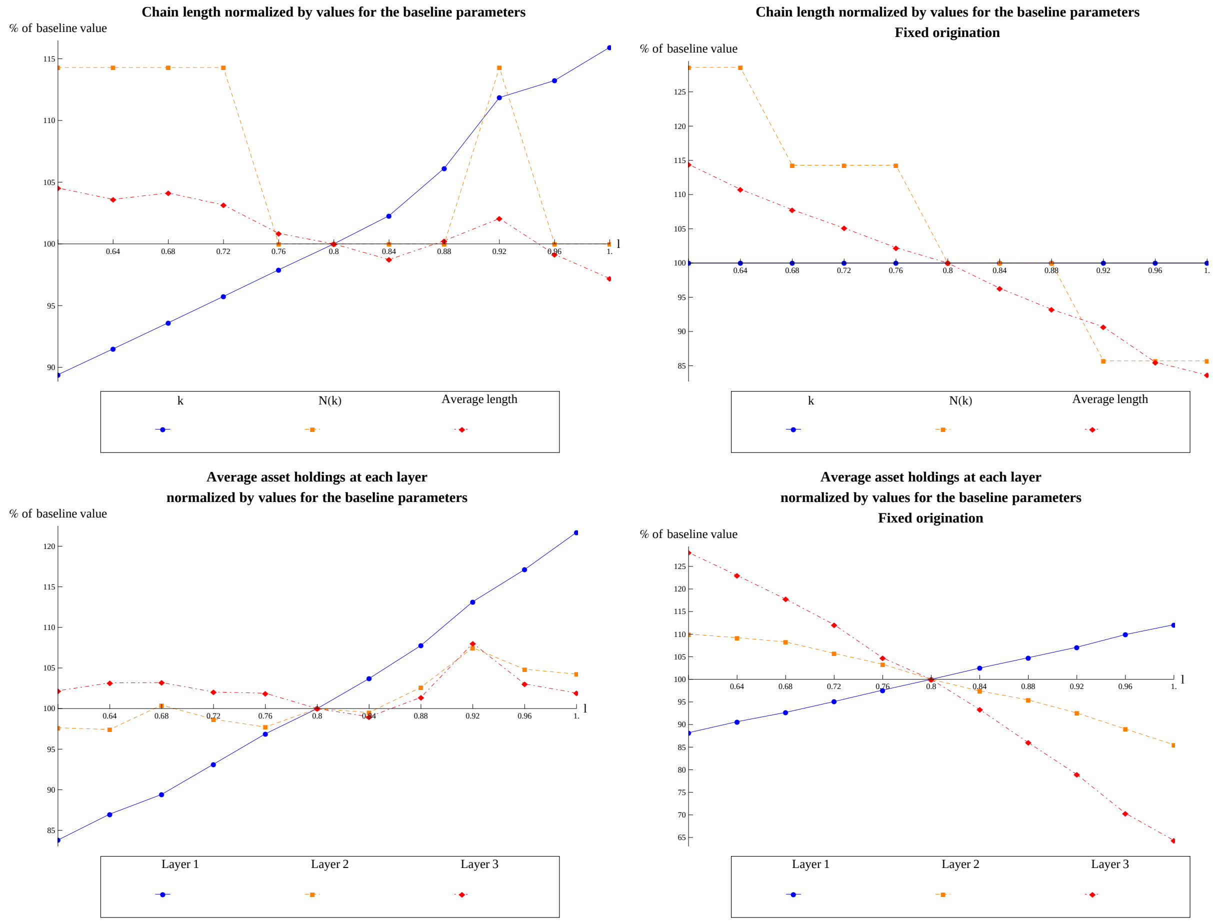

Figure 6: Impact of the collateral value on chain length and asset holdings.

Reading (e.g. top left panel): when the collateral value ℓ is 0.68 instead of its baseline value of 0.8, k, $N(k)$ and the averaged length of the realized chain are to about 93%, 104% and 114% of their values under the baseline parameters, respectively.

Layer	v_0	p	v	Holdings	ω	Ω	B	p_a	p_r	p_c
Orig.	1.027	2.71	0.384	0.715	0	0.130	0.390	1	0	0
1	0.384	2.99	0.203	0.222	0.315	0.121	0.274	0.794	0.1867	0.0193
2	0.202	3.26	0.097	0.144	0.314	0.103	0.164	0.468	0.1431	0.1829
3	0.097	3.41	0.049	0.081	0.310	0.080	0.097	0.1436	0.0784	0.246
4	0.049	3.45	0.032	0.045	0.306	0.062	0.067	0.0178	0.0211	0.1047
5	0.030	3.5	0.014	0.030	0.304	0.037	0.032	0.0009	0.0024	0.0145
6	0.012	$n.a.$	$n.a.$	0.012	0.203	$n.a.$	$n.a.$	$n.a.$	0.0001	0.0008

Table 1: Simulation results. Empirical averages under the baseline parameters.

Param./Effect	Trading	Financing needs	Origination
ρ	0	0	$+$
ℓ	0	$-$	$+^\dagger$
r	0	0	$-$
d	$-^*$	0	$+$
q	$-^*$	0	$+$
Asym. info.	$+^*$	0	$-$

Table 2: Direction of the different effects on the size of the targeted chain.

The effects denoted with a $*$ are based on the study of a lower bound on chain size only. The effect denoted with a $\dagger$ is based on the assumption that B^* is concave.

B Appendix-Proof of Theorem 1

We prove Theorem 1 in three steps. We start by redefining strategy and benefit functions in the auxiliary games $\mathcal{G}_n$. We then define recursively a function denoted $B_n^*(y)$ and show a number of its properties in Lemma 1. Then we show in Proposition 2 that the equilibrium benefit function for intermediaries accepting offers coincides with $B_n^*(y)$. We conclude by showing how the properties of $B_n^*(y)$ shown in Lemma 1 imply the theorem.

B.1 Definition of benefit and strategy functions in the games $\mathcal{G}_n$

In what follows, it will be convenient to define the *transactions value* of an offer (p, v) made by an intermediary with financing needs y:

$$T(y; p, v) = \frac{1}{r}\big(ry - (\rho - p)v - r[y - (p - \ell)v]^+\big)$$

Assume intermediary I is the first player in the game $\mathcal{G}_{n+1}$, faces the offer (p_0, v_0) and has positive financing needs y (the case of null y is trivial). Notice that I could be the originator O having to finance the same amount. If I proposes (p, v), then a receiver R plays the game $\mathcal{G}_n$ as the first player, faces (p, v) and has cash ω, giving financing needs $y_R = [(p - \ell)v - \omega]^+$. Let us denote by $B_n(p, v, y_R)$ R's equilibrium expected benefit by making a proposal in $\mathcal{G}_n$. R's expected profit if he accepts (p, v) is

$$\Pi_n(p, v, y_R) = (\rho - p)v - ry_R + rB_n(p, v, y_R) \text{ with } y_R = [(p - \ell)v - \omega]^+. \tag{27}$$

Denote $W_{n+1}(p, v)$ the associated threshold, that is the minimum cash for R that ensures non-negative profit to R:

$$W_{n+1}(p, v) = \inf\{\omega \text{ such that } \Pi_n(p, v, max[(p_0 - \ell)v_0 - \omega, 0]) \geq 0\}.$$

In $\mathcal{G}_{n+1}$, I expects the acceptance probability to be given by $H(W_{n+1}(p, v))$, so Property 1 holds. Furthermore, the threshold is always positive. If that's not the case, then I's benefit is equal to $H(0)T(y; , p, v)$. $W_{n+1}(p, v) = 0$ implies that $rB_n((p - \ell)v) = -(\rho - p)v + r((p - \ell)v)$.

If $y \geq (p - \ell)v$ we get $H(0)T(y; p, v) \leq H(0)B_n(p, v, (p - \ell)v)$. Since I can get $B_n(y) \geq B_n(p, v, (p - \ell)v)$ by playing the strategy chosen by an intermediary with financing needs y in $\mathcal{G}_n$ this is surely not optimal.

If $y < (p - \ell)v$ we have $H(0)T(y; p, v) = H(0)(B_n(p, v, (p - \ell)v) + (y - (p - \ell)v))$. I cannot directly get $B_n(p, v, (p - \ell)v)$ as his financing needs are too low. Notice however that for any y and any dy we have $B_n(p; v, y + dy) \leq B_n(p, v, y) + H(0)dy$ as the best an intermediary can hope for with extra financing costs dy is to have the full amount covered with the maximal probability $H(0)$. This gives us that necessarily $B_n(p, v, (p - \ell)v) \leq B_n(p, v, y) + H(0)((p - \ell)v - y)$. Thus $H(0)T(y; , p, v) \leq H(0)B_n(p, v, y) - ((p - \ell)v - y)(1 - H(0)) < H(0)B_n(p, v, y)$. Since by definition $v \leq v_0$, I can at least get $B_n(p, v, y)$ and we have a contradiction.

The maximal benefit that I can expect is thus

$$B_{n+1}(p_0, v_0, y) = \max_{p, v, p \leq \rho, v \leq v_0} H(W_n(p, v))T(y; p, v) \tag{28}$$

This defines the equilibrium functions B_n recursively.

B.2 Definition and properties of B_n^*

We want to show that the functions B_n actually have a simple form. Starting with B_1^*, let us define recursively the functions B_n^* for $n \geq 1$ by:

$$B_n^*(y) = \max_{\omega \leq y} H(\omega)(\omega + B_{n-1}^*(y - \omega)) \tag{29}$$

The next lemma collects useful properties on the functions B_n^*:

Lemma 1. *The functions B_n^* satisfy for all $n \geq 1$:*

1. *Monotony and contraction: for $y' < y$, $0 \leq B_n^*(y) - B_n^*(y') \leq H(0)(y - y')$.*
2. *Let $b_n^{\max}$ be the maximum value of B_n^* and ω_{n+1} the maximizer of $H(\omega)(\omega + b_n^{\max})$. We have*

$$\omega_{n+1} > 0 \text{ and } b_{n+1}^{\max} = H(\omega_{n+1})(\omega_{n+1} + b_n^{\max}) \tag{30}$$

The sequence $b_n^{\max}$ is increasing with limit $b^{\max}$ and the sequence ω_n is decreasing with limit $\omega^{\max}$ where $(b^{\max}, \omega^{\max})$ satisfy:

$$b^{\max} = H(\omega^{\max})(\omega^{\max} + b^{\max}) \text{ and } \frac{H'(\omega^{\max})}{H(\omega^{\max})} + \frac{1}{\omega^{\max} + b^{\max}} = 0. \tag{31}$$

Define $\overline{y}_1 = \omega^*$ and for any $n \geq 1$, $\overline{y}_{n+1} = \overline{y}_n + \omega_{n+1}$. $B_n^*(y)$ strictly increases for $y < \overline{y}_n$ and is constant for $y \geq \overline{y}_n$. B_n^* is differentiable at $\overline{y}_n$ with a null derivative.

3. *There exists $\underline{y}_n < \overline{y}_n$ such that $B_n^*(y) = B_{n-1}^*(y)$ for $y \leq \underline{y}_n$ and $B_n^*(y) > B_{n-1}^*(y)$ for $y > \underline{y}_n$. Moreover $\underline{y}_{n+1} > \underline{y}_n$ and the sequence $\underline{y}_n$ goes to $+\infty$.*

Note that the contraction property implies the continuity of B_n^*.

Proof: *1.* The monotony is straightforward by induction. Then B_1^* trivially satisfies the contraction inequality. Let B_{n-1}^* satisfy it and pick $y' < y$. Denote $\hat{\omega}$ a maximizer of $H(\omega)(\omega + B_{n-1}^*(y - \omega))$ and $\hat{\omega}'$ a maximizer of the same expression with y'.

Assume $\omega \leq y'$. Then by definition $B_n^*(y') \geq H(\omega)(\omega + B_{n-1}^*(y' - \omega))$. Thus

$$0 \leq B_n^*(y) - B_n^*(y') \leq H(\omega)(B_{n-1}^*(y - \omega) - B_{n-1}^*(y' - \omega)) \leq H(0)^2(y - y')$$

Assume $\omega > y'$. Using both inequalities $B_n^*(y') \geq H(y')y'$ and $H(\omega) \leq H(y')$, we obtain

$$0 \leq B_n^*(y) - B_n^*(y') \leq H(y')(\omega + B_{n-1}^*(y - \omega) - y').$$

By the contraction property of B_{n-1}^*, $B_{n-1}^*(y - \omega) \leq H(0)(y - \omega)$ since $B_{n-1}^*(0) = 0$. This finally gives $0 \leq B_n^*(y) - B_n^*(y') \leq H(0)(y - y')$.

2. Consider any $b \geq 0$. Under log-concavity of H, the log of $H(\omega)(\omega + b)$ is concave with a derivative equal to

$$\frac{H'(\omega)}{H(\omega)} + \frac{1}{\omega + b}. \tag{32}$$

It follows that the maximizer of $\max_{\omega \geq 0} H(\omega)(\omega + b)$ is unique, and positive if $\frac{H'(0)}{H(0)} + \frac{1}{b} > 0$. Defining $b_n^{\max}$ and ω_n as in the Lemma, we prove by induction that $\frac{H'(0)}{H(0)} + \frac{1}{b_n^{\max}} > 0$ and thus that $\omega_{n+1} > 0$ for any n.

When there is a single layer, $b = 0$ and the assumption is satisfied with a maximum reached at ω^*. If the induction assumption is valid for a given level n, then $\omega_{n+1} > 0$ and is characterized by:

$$\frac{H'(\omega_{n+1})}{H(\omega_{n+1})} + \frac{1}{\omega_{n+1} + b_n^{\max}} = 0. \tag{33}$$

We want to show that $\frac{H'(0)}{H(0)} + \frac{1}{b_{n+1}^{\max}} > 0$. The log concavity of H implies $\frac{H'(0)}{H(0)} \geq \frac{H'(\omega_{n+1})}{H(\omega_{n+1})}$. Using the first-order condition (33), we have

$$\frac{H'(0)}{H(0)} + \frac{1}{b_{n+1}^{\max}} \geq \frac{1}{b_{n+1}^{\max}} - \frac{1}{\omega_{n+1} + b_n^{\max}}$$

which is positive given the definition of $b_{n+1}^{\max}$. This proves that ω_n is positive for any n and the sequence $(b_n^{\max}, \omega_n)$ is characterized by

$$b_{n+1}^{\max} = H(\omega_{n+1})(\omega_{n+1} + b_n^{\max}) \text{ and } \frac{H'(\omega_{n+1})}{H(\omega_{n+1})} + \frac{1}{\omega_{n+1} + b_n^{\max}} = 0. \tag{34}$$

It is trivial to show that $b_n^{\max}$ is increasing, ω_n decreasing and they converge to $(b^{\max}, \omega^{\max})$.

That $B_{n+1}^*(y)$ is increasing in y and constant for y larger than some threshold $\overline{y}_{n+1}$ is easily shown by induction. Furthermore, $B_{n+1}(y) = H(\omega)(\omega + b_n^{\max})$ for $y - \omega > \overline{y}_n$. Hence $b_{n+1}^{\max}$ is the maximum of B_{n+1}^*, reached at $\overline{y}_{n+1} = \overline{y}_n + \omega_{n+1}$ and B_{n+1}^* is constant for $y > \overline{y}_{n+1}$. By the induction assumption B_n^* is differentiable at $\overline{y}_n$. Hence the envelope theorem applies: B_{n+1}^* is differentiable at $\overline{y}_{n+1}$ and $B_{n+1}^{*\prime}(\overline{y}_{n+1}) = H(\omega_{n+1}) B_n^{*\prime}(\overline{y}_n) = 0$.

3. At an optimal target $\omega \in \Omega_n(y)$: $B_{n+1}^*(y) = H(\omega)(\omega + B_n^*(y - \omega))$. The following holds:

$$B_n^*(y) \leq B_{n+1}^*(y) \leq H(0)(\omega + B_n^*(y - \omega)) \leq H(0)(\omega + B_n^*(y)), \text{ at } \omega = \Omega_n(y)$$

The first inequality holds because a player with needs y playing $\mathcal{G}_{n+1}$ can always play as in $\mathcal{G}_n$ hence secure at least $B_n^*(y)$. The second inequality holds because H is non-increasing and B_n^* non-decreasing.

This implies $\omega \geq B_n^*(y)(1 - H(0))/H(0)$. For $y \geq \omega^* = \overline{y}_1$, $B_n^*(y) \geq H(\omega^*)\omega^*$ gives a lower bound to any target: $\omega_{min} = H(\omega^*)\omega^*(1 - H(0))/H(0)$.

An intermediary playing $\mathcal{G}_{n+1}$ with financing needs equal to $y = \underline{y}_n + \omega_{min}$ thus chooses $\omega \geq$

ω_{min}, so that his target has financing needs equal to $y-\omega \leq \underline{y}_n$. Thus $B^*_{n+1}(y) = H(\omega)(\omega + B^*_n(y-\omega)) = H(\omega)(\omega + B^*_{n-1}(y-\omega))$. The intermediary would take the same decision if he were playing $\mathcal{G}_n$, which shows that $\underline{y}_{n+1} \geq \underline{y}_n + \omega_{min}$. Thus the sequence $\underline{y}_n$ strictly increases and goes to infinity.
■

B.3 Proof that in equilibrium $B_n(p_0, v_0, y) = B^*_n(y)$ - Lemmas

We now prove the following proposition, which says that the benefit of an intermediary with financing needs is precisely equal in equilibrium to the quantity $B^*_n(y)$ just defined.

Proposition 2. *For any $n \geq 1$, let intermediary I accept offer (p_0, v_0) in $\mathcal{G}_n$ and have positive financing needs y. I's expected benefit $B_n(p_0, v_0, y)$ depends only on y and not on p_0, v_0 separately, and is equal to $B^*_n(y)$.*

Furthermore, let $\Omega_{n+1}(y)$ denote a maximizer (surely strictly positive) of $B^*_{n+1}(y) = \max_{\omega \leq y} H(\omega)(\omega + B^*_n(y-\omega))$. If I accepts an offer and has $y < \overline{y}_{n+1}$, I's optimal offer is uniquely characterized by the two equations

$$(p-\ell)v = y \text{ and } (\rho - p)v = r(y - \Omega_{n+1}(y) - B^*_n(y - \Omega_{n+1}(y))). \tag{35}$$

The proof relies on the lemmas 2 to 4. In these lemmas, intermediary I is the first player in the game $\mathcal{G}_{n+1}$ so I's expects the receiver's benefit to be given by B^*_n. I faces offer (p_0, v_0) and has positive financing needs y. We first show that an optimal offer never over-finances the needs.

Lemma 2. *Let I accept (p_0, v_0) and have positive financing needs y. I makes an offer (p, v) that has a cash value not greater than y.*

Proof. By contradiction let us assume $(p-\ell)v > y$. I's transaction value satisfies $rT(y;p,v) = ry - (\rho - p)v$. Let us distinguish two cases.

In the first case, $p = \rho$; I's transaction value is equal to y for offers (ρ, v') with v' close to v as long as $y < (\rho - \ell)v'$ and provides the benefit $H(W_{n+1}(\rho, v'))y$ to I. We know that the threshold $W_{n+1}(\rho, v')$ is equal to the cash value $(\rho - \ell)v'$ (Point 3 of Property 1). Therefore, decreasing v allows to increase the chances of success, hence I's expected benefit. A contradiction.

In the second case, $p < \rho$. For offers (p', v') close to (p, v) so that the cash value still exceeds y, $y < (p' - \ell)v'$, I's transaction value satisfies $rT(y; p', v') = ry - (\rho - p')v'$ and I's expected benefit is

$$H(W_{n+1}(p', v'))(ry - (\rho - p')v').$$

We show that the offer can be adjusted in such a way that $(\rho - p')v'$ is constant and the acceptance probability increases, that is $W_{n+1}(p', v')$ decreases.

By the induction assumption, the benefit of a receiver R accepting the offer and with financing needs y_R is given by $B_n^*(y_R)$ independently of the offer. Recall that at the threshold of an offer with $p < \rho$, financing needs are positive, $y_R > 0$ and R's profit is null (from Proposition **??**). Applying these properties to offers (p, v) and (p', v') and $y_R = (p - \ell)v - W_{n+1}(p, v)$ and $y'_R = (p' - \ell)v' - W_{n+1}(p', v')$, we have

$$(\rho - p)v - ry_R + rB_n^*(y_R) = 0 \text{ and } (\rho - p')v' - ry'_R + rB_n^*(y'_R) = 0.$$

Choose $v' < v$ and $p' > p$ so as to keep $(\rho - p)v = (\rho - p')v'$ (this is possible since $p < \rho$); surely $(p - \ell)v > (p' - \ell)v'$ and for v' close enough to v, $(p' - \ell)v' > y$; the above equalities imply $y'_R - B_n^*(y'_R) = y_R - B_n^*(y_R)$. Hence $y'_R = y_R$ since, by the induction assumption, B_n^* is a contraction with a constant less than $H(0) < 1$. Hence $(p - \ell)v - W_{n+1}(p, v) = (p' - \ell)v' - W_{n+1}(p', v')$. Using $(p - \ell)v > (p' - \ell)v'$, it follows that $W_{n+1}(p, v) > W_{n+1}(p', v')$, the desired inequality. ∎

Lemma 3. *I's benefit from offering (p, v) satisfies*

$$H(W_{n+1}(p, v))T(y; p, v) = H(W_{n+1}(p, v))(W_{n+1}(p, v) + B_n^*((p - \ell)v - W_{n+1}(p, v)) \tag{36}$$

$$\leq H(W_{n+1}(p, v))(W_{n+1}(p, v) + B_n^*(y - W_{n+1}(p, v)). \tag{37}$$

It follows that $B_{n+1}(p_0, v_0, y) \leq B_{n+1}^*(y)$.

Proof. From Lemma 2, I's optimal offer is less than his financing needs, which yields a transaction value of the form

$$rT(y; p, v) = -(\rho - p)v + r(p - \ell)v = (1 + r)(p - \tau)v. \tag{38}$$

So R's profit with cash ω and positive needs y_R writes

$$\Pi_n(p, v, y_R) = -rT(y; p, v) + r\omega + rB_n^*((p - \ell)v - \omega). \tag{39}$$

At the target, this profit is null: thus $T(y; p, v) = r\omega + rB_n^*((p - \ell)v - \omega)$ at $\omega = W_{n+1}(p, v)$, which gives (36). The inequality (37) follows since $(p - \ell)v \leq y$ and B_n^* is non-decreasing.

Using the definition (29) of $B_{n+1}^*(y)$, the term on (37) is bounded above by $B_{n+1}^*(y)$. Thus the maximum of $H(W_{n+1}(p, v))T(y; p, v)$ over the feasible (p, v), $B_{n+1}(p_0, v_0, y)$, is not larger than $B_{n+1}^*(y)$.

∎

Lemma 4. *I's optimal offer exactly covers his financing needs when $y \leq \overline{y}_{n+1}$.*

Proof. We show that for y small enough, $(p-\ell)v < y$ is excluded. The basic intuition is easy: assuming the receiver accepts the offer, the transaction value that accrues to I is withdrawn from R's profit as can be seen from (38) and (39); by keeping this transaction value constant and increasing R's needs, R can draw more benefit from his partner intermediaries if his financing needs are low enough, lower than $\overline{y}_n$; this yields the possibility for an improvement for both. At the opposite, if R's financing needs are larger than $\overline{y}_n$, the situation is akin to a zero-sum game, and increasing the cash value and R's financing needs is not beneficial. Let us prove this formally.

Step 1: *an optimal offer (p, v) has $(p-\ell)v < y$ only if the targeted R financing needs are above $\overline{y}_n$: $(p-\ell)v - W_{n+1}(p, v) \geq \overline{y}_n$.*

Start with an offer (p, v) with $(p-\ell)v < y$. Increase the price and decrease the volume so as to leave $(p-\tau)v$ constant. The transaction value is not affected. The financing needs of the target receiver are equal to $y_R = (p-\ell)v - \omega_R$ and thus increase. If $y_R < \overline{y}_n$, the receiver's benefit $B_n^*(y_R)$ increases (see (39)). Thus the receiver at the threshold $W_{n+1}(p, v)$ makes a positive profit. Since $W_{n+1}(p, v) > 0$ (as proved in B.1) there are some receivers with lower cash who accept the new offer and the acceptance probability is increased: (p, v) cannot be optimal. ∎

Step 2: *an optimal offer (p, v) has $(p-\ell)v < y$ only if $y \geq \overline{y}_{n+1}$ where $\overline{y}_{n+1} = \overline{y}_n + \omega_{n+1}$.*

From Step 1 $y_R = (p-\ell)v - W_{n+1}(p, v) \geq \overline{y}_n$. Surely $p < \rho$ because $y_R > 0$. We show that $W_{n+1}(p, v) \geq \omega_{n+1}$. This will give the result.

By contradiction assume $W_{n+1}(p, v) < \omega_{n+1}$. Consider an increase in p to p' small enough so that $p' < \rho$ and $(p'-\ell)v < y$. The threshold increases: it satisfies

$$(\rho - p)v = r(y_R - B_n^*(y_R)) \text{ where } y_R = (p-\ell)v - W_{n+1}(p, v))$$

By the contraction property, $y_R - B_n^*(y_R)$ is increasing in y_R. Hence, an increase in p decreases y_R: the increase in $(p-\ell)v$ is more than compensated by the increase in $W_{n+1}(p, v)$: $W_{n+1}(p', v) > W_{n+1}(p, v)$ and $y_R > y'_R$. From (36) in Lemma 3, I's expected benefit writes

$$H(W_{n+1}(p', v))(W_{n+1}(p', v) + B_n^*(y'_R)).$$

We treat first the easy case where $y_R = (p-\ell)v - W_{n+1}(p, v) > \overline{y}_n$. For a marginal increase in p', $y'_R > \overline{y}_n$. Thus I's expected benefit is given by $H(W_{n+1}(p', v))(W_{n+1}(p', v) + b_n^{\max})$. Recall the function $H(\omega)(\omega + b_n^{\max})$ increases for $\omega < \omega_{n+1}$: the increase in p increases $W_{n+1}(p, v)$ hence I's profit, a contradiction.

Assume now $y_R = \overline{y}_n$. The marginal change contemplated above makes y'_R fall below $\overline{y}_n$, and I's expected benefit is given by $H(W_{n+1}(p, v))(W_{n+1}(p, v) + B_n^*(y'_R))$. $B_n^*(y'_R) - B_n^*(\overline{y}_n)$ is negligible because B_n^* has a null derivative at $\overline{y}_n = y_R$. I's expected benefit is thus still increasing at p if

$W_{n+1}(p,v) < \omega_{n+1}$. ■

B.4 Proof that in equilibrium $B_n(p_0, v_0, y) = B_n^*(y)$ - Conclusion

We can now conclude the proof of Proposition 2. It is first easy to check that the Proposition is valid for $n = 1$. Assume it is valid up to some $n \geq 1$ and let I play $\mathcal{G}_{n+1}$ and accept an offer (p_0, v_0).

We know from Lemma 3 that $B_{n+1}^*(y)$ is an upper bound on I's benefit. We show that I's optimal offer reaches the benefit $B_{n+1}^*(y)$. Let $\Omega_{n+1}(y)$ denote a maximizer (surely strictly positive) of $B_{n+1}^*(y) = \max_{\omega \leq y} H(\omega)(\omega + B_n^*(y - \omega))$.

We consider two cases. In the first case the optimal offer is unique.

Case 1 : $y < \overline{y}_{n+1}$. From Lemma 4, the offer has surely $(p - \ell)v = y$. To achieve $B_{n+1}^*(y)$ involves choosing (p, v) that satisfies $W_{n+1}(p,v) = \Omega_{n+1}(y)$. This implies that I's optimal offer must satisfy the two equations

$$(p - \ell)v = y \text{ and } (\rho - p)v = r(y - \Omega_{n+1}(y) - B_n^*(y - \Omega_{n+1}(y)). \tag{40}$$

A pair (p, v) satisfying (40) is unique, has a price $p < \rho$ and a positive value for v. To see this, remember that $z - B_n^*(z) > 0$ for a positive z. Applied to $z = y - \Omega_{n+1}(y)$, this gives that both $(p - \ell)v$ and $(\rho - p)v$ are positive, which implies $p < \rho$ and a positive value for v.

Thus, $B_{n+1}^*(y)$ can be achieved by a feasible offer if (and only if) $v \leq v_0$. We show that it is true if I accepts the offer. I's expected profit π when offering (p, v) is

$$\pi = (\rho - p_0)v_0 - ry + H(\Omega_{n+1}(y))(ry - (\rho - p)v).$$

I accepts (p_0, v_0) only if π is non-negative. This implies $(\rho - p_0)v_0 > (\rho - p)v$: Otherwise, $\pi \leq \big((\rho - p)v - ry\big)(1 - H(\Omega_{n+1}(y))$, but $ry - (\rho - p)v = r(\Omega_{n+1}(y) + B_n^*(y - \Omega_{n+1}(y))$ is positive (the second equality in (40)), a contradiction. So we have $(\rho - p_0)v_0 > (\rho - p)v$ and $(p - \ell)v = y = (p_0 - \ell)v_0 - \omega_0$. If $v \geq v_0$, the first inequality yields $p_0 < p$ and the second $p_0 \geq p$: a contradiction.

Case 2 : $y \geq \overline{y}_{n+1}$. We have $\Omega_{n+1}(y) = \beta_{n+1}$, and $B_n^*(y - \Omega_{n+1}(y)) = \overline{b}_n$. The same argument as in Step 1 can be used. Hence the offer (p, v) that satisfies $(p - \ell)v = y$ and $(\rho - p)v = r(y - \beta_{n+1} - \overline{b}_n))$ is feasible for I and yields I's maximal benefit. The only difference with case 1 is that there are other offers that reach the same benefit : decrease p so as to keep $(p - \tau)v$ constant and the financing needs of R larger than $\overline{y}_n$. ■

C Appendix-Other proofs

C.1 Proof of Corollary 1

We first show that $Z(y) = y - \Omega(y)$ is increasing. Take two intermediaries I and I' with levels of financing needs y and y', $y' > y$. Assume I' chooses $\Omega(y') = \Omega(y) + (y' - y)$, so that $Z(y) = Z(y')$. We will show that I' has an incentive to decrease $\Omega(y')$ (this reasoning assumes profit is concave, which is true at least locally).

Assume $y - \Omega(y) > 0$. From the first-order condition associated to (11) we deduce:

$$\begin{aligned}
\frac{\partial B^*(y')}{\partial \omega}|_{\omega=\Omega(y')} &= H'(\Omega(y'))(\Omega(y') + B^*(y' - \Omega(y'))) + H(\Omega(y'))(1 - B^{*\prime}(y' - \Omega(y'))) \\
&= H'(\Omega(y'))(\Omega(y') + B^*(y - \Omega(y))) - \frac{H(\Omega(y'))H'(\Omega(y))}{H(\Omega(y))}(\Omega(y) + B^*(y - \Omega(y))) \\
&= H(\Omega(y'))\left[\left(\frac{H'(\Omega(y'))}{H(\Omega(y'))} - \frac{H'(\Omega(y))}{H(\Omega(y))}\right)(\Omega(y) + B^*(y - \Omega(y))) + \frac{H'(\Omega(y'))}{H(\Omega(y'))}(y' - y)\right]
\end{aligned}$$

Since $y' > y$ and $H'(\Omega(y')) \leq 0$ the second term is negative. When H is log-concave, since $\Omega(y') = \Omega(y) + y' - y > \Omega(y)$ we have $(H'(\Omega(y'))/H(\Omega(y'))) - (H'(\Omega(y))/H(\Omega(y))) \leq 0$, and thus the derivative of I''s profit when he chooses $\Omega(y') = \Omega(y) + y' - y$ is negative, hence the optimal $\Omega(y')$ needs to be smaller. Finally, if $y - \omega = 0$, since $y' - \omega'$ must be positive necessarily $y' - \omega' \geq y - \omega$. ▪

C.2 Proof of Proposition 1

The originator's profit if he chooses k is equal to:

$$\Pi_O(k) = \rho k - C(k) - r(C(k) - \ell k - B^*(C(k) - \ell k))$$

The derivative of the profit is

$$\frac{\partial \Pi_O}{\partial k} = \rho - C'(k) - r(1 - B^{*\prime}(C(k) - \ell k)(C'(k) - \ell).$$

The first-order condition gives that the optimal k satisfies (21). The second derivative is negative at the optimal k. The cross derivative $\frac{\partial^2 \Pi_O}{\partial k \partial \rho}$ is equal to 1. The monotonicity of the optimal k with respect to ρ follows.

$p = \rho$ is optimal if $y(k) = C(k) - \ell k$ is less than $\underline{y}_1$. An optimal k must necessarily be such that $C'(k) > \ell$, so that increasing ρ will increase the chosen k until $\underline{y}_1$ is reached. This shows that surely for ρ large enough the optimal k will give $y(k) > \underline{y}_1$ and thus an offered price $p < \rho$.

The derivative $\frac{\partial \Pi_O}{\partial k}$ decreases with r hence the optimal investment is non-increasing with r. Similarly, when B^* is concave, the derivative increases with ℓ hence the optimal investment is non-decreasing in ℓ. ▪

C.3 Proof of Implication 4

Considering the originator's optimal choice k, we have:

$$\frac{\partial \Pi_O^2}{\partial k \partial \ell} = r[1 - B^{*\prime}(y(k)) - B^{*\prime\prime}(y(k))(c-\ell)k], \ \frac{\partial \Pi_O^2}{\partial^2 k} = rB^{*\prime\prime}(y(k))(c-\ell)^2$$

At the optimal solution the second-order condition is satisfied and thus $B^{*\prime\prime}(y(k)) \leq 0$. This gives:

$$\frac{\partial k}{\partial \ell} = \frac{k}{(c-\ell)} - \frac{1 - B^{*\prime}(y(k))}{B^{*\prime\prime}(y(k))(c-\ell)^2} > 0.$$

Thus the originated volume increases when ℓ is higher. Moreover,

$$\frac{\partial y(k)}{\partial \ell} = (c-\ell)\frac{\partial k}{\partial \ell} - k = -\frac{1 - B^{*\prime}(y(k))}{B^{*\prime\prime}(y(k))(c-\ell)} > 0$$

This proves that k and $y(k)$ increase when ℓ increases.

Let us consider now the targeted chain. Since $Z(y)$ increases with y and is independent of ℓ, the financing needs of the targeted intermediaries are $Z(y(k))$, $Z^2(y(k))$, $Z^3(y(k))$ etc.. all increase. This implies that the length of the targeted chain does not decrease. The volume sold depends on ℓ, so write it as $V(y, \ell)$. V is increasing in y (Corollary 1) and ℓ (Implication 1). Thus the first intermediary's offered volume $V(Z(y(k)), \ell)$ increases with ℓ. The same argument applies to the volume $V(Z^t(y(k)), \ell)$ at each level. ■

C.4 Proof of Implication 5

It is enough to show that for any ω the quantity $H'(\omega)/H(\omega)$ is increasing in q and d. We have:

$$\frac{H'(\omega)}{H(\omega)} = \frac{-dqg(\omega)(1-q+qG(\omega))^{d-1}}{1-(1-q+qG(\omega))^d}$$

which gives:

$$\begin{aligned} \frac{\partial H'(\omega)/H(\omega)}{\partial q} &= \frac{dg(\omega)(1-q+qG(\omega))^{d-2}}{H(\omega)^2}\phi_q(G(\omega)) \\ \text{with } \phi_q(G(\omega)) &= -1 + (1-q+qG(\omega))^d + dq(1-G(\omega)) \end{aligned}$$

$G(\omega)$ is necessarily between 0 and 1. We have $\phi_q(1) = 0$ and ϕ_q is decreasing in $G(\omega)$, so that $\phi_q(G(\omega))$ is always positive, which shows that $H'(\omega)/H(\omega)$ is increasing in q. Similarly for d:

$$\begin{aligned} \frac{\partial H'(\omega)/H(\omega)}{\partial q} &= \frac{qg(\omega)(1-q+qG(\omega))^{d-1}}{H(\omega)^2}\phi_d(G(\omega)) \\ \text{with } \phi_d(G(\omega)) &= -1 + (1-q+qG(\omega))^d - d\ln(1-q+qG(\omega)) \end{aligned}$$

We find again that $\phi_d(1) = 0$ and ϕ_d is decreasing in $G(\omega)$, showing that $H'(\omega)/H(\omega)$ is increasing in d and concluding the proof. ■

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